MATH2022 Week 08

Worksheet

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Q1/ Consider the linear transformation

$f : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$f(x,y) = (2x+y,\ 2x-y,\ y-x).$$

Find

$$f(1,0) = \boxed{}$$

$$f(0,1) = \boxed{}$$

and the matrix M representing f:

$$M =$$

Check that $M\begin{bmatrix} x \\ y \end{bmatrix} = (f(x,y))^T$:

$$M\begin{bmatrix} x \\ y \end{bmatrix} =$$

Q2/ Consider $f: \mathbb{R}^4 \to \mathbb{R}^3$ where

$$f(x,y,z,w) = (y-z+w,\ x-y+z,\ 2x-3w).$$

Find

$f(1,0,0,0) = \boxed{}$

$f(0,1,0,0) = \boxed{}$

$f(0,0,1,0) = \boxed{}$

$f(0,0,0,1) = \boxed{}$

and the matrix M representing f:

$M =$

Check that $M\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = (f(x,y,z,w))^T$:

$M\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} =$

Q3/ Consider $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ represented respectively by

$$M_f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_g = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad M_h = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Find the following rules :

$f(x,y) =$

$g(x,y) =$

$h(x,y) =$

Find the following composite rules directly :

$(f \circ g)(x,y) = f(g(x,y)) =$

$(g \circ f)(x,y) = g(f(x,y)) =$

$(f \circ h)(x,y) =$

$(h \circ h)(x,y) =$

Find the following matrix products:

$$M_f M_g = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$$

$$M_g M_f = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} =$$

$$M_f M_h = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} =$$

$$M_h^2 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} =$$

Check that you obtain the transposes of the previous rules, using matrix multiplication:

$$M_f M_g \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$M_g M_f \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$M_f M_h \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$M_h^2 \begin{bmatrix} x \\ y \end{bmatrix} =$$

Q4/ Find all powers of 3 in $\mathbb{Z}_{13}$:

$3^1 = 3$, $3^2 =$

Thus

$$\langle 3 \rangle = \boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$$

Find all powers of 2 in $\mathbb{Z}_{13}$:

$2^1 = 2$, $2^2 =$

Thus

$$\langle 2 \rangle = \boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$$

True or False:

$\mathbb{Z}_{13} \setminus \{0\}$ is a cyclic group? T F

Find the rule for an isomorphism from $\mathbb{Z}_{13} \setminus \{0\}$ to $(\mathbb{Z}_{12}, +)$:

$$2^i \mapsto \boxed{\quad} \quad \text{for } i = 1, \dots, 12.$$

Q5/ The symbols $\langle \ \rangle$ need to be read in context. In this exercise, generation is with respect to addition $+$.

True or False:

(a) $\mathbb{Z}_6 = \langle 1 \rangle$ T F

(b) $\mathbb{Z}_6 = \langle 2 \rangle$ T F

(c) $\mathbb{Z}_6$ is cyclic. T F

(d) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,1) \rangle$ T F

(e) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,2) \rangle$ T F

(f) $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic. T F

(g) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle (1,1) \rangle$ T F

(h) $\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle (1,2) \rangle$ T F

(i) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic. T F

(j) $\mathbb{Z}_3 \times \mathbb{Z}_4$ is cyclic. T F

(k) $\mathbb{Z}_4 \times \mathbb{Z}_6$ is cyclic. T F

(l) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ is cyclic. T F

Q6/ Consider the 15-puzzle:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Consider the following configuration:

1	5	8	10
11	2	6	9
14	12	3	7
	15	13	4

Write down the associated permutation where the blank square is labelled 16 :

Is it even or odd?

Is the configuration possible? Y N

Q7/ Let $G = \langle \alpha, \beta \rangle$ be the group of symmetries of a regular n-gon where $n \geq 3$ and

$\alpha =$ rotation $2\pi/n$,

$\beta =$ reflection in a fixed axis of symmetry.

Then

$$G = \{ \underbrace{1, \alpha, \alpha^2, \dots, \alpha^{n-1}}_{\text{rotations}}, \underbrace{\beta, \alpha\beta, \dots, \alpha^{n-1}\beta}_{\text{reflections}} \}$$

and $\boxed{\alpha^n = \beta^2 = 1, \ \alpha\beta = \beta\alpha^{-1}, \ \beta\alpha = \alpha^{-1}\beta}$

Put

$$\gamma = \alpha^4 \beta^3 \alpha^{-2} \beta \alpha^5 \beta \alpha^{-3}.$$

Simplify γ in the following cases:

(a) $n = 5$:

(b) $n = 6$:

(c) $n = 7$: