MATH2022 Week 10

Worksheet Solutions

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Q1/ What is a **basis** for a vector space?

> A **basis** is a linearly independent spanning set.

What is the **dimension** of a vector space?

> The **dimension** of a vector space is the size of any basis.

True or False:

(i) $\{(1,2),(2,1)\}$ is a basis for $\mathbb{R}^2$. T F

(ii) $\{(1,2),(2,1)\}$ is a basis for $\mathbb{Z}_5^2$. T F

(iii) $\{(1,2),(2,1)\}$ is a basis for $\mathbb{Z}_3^2$. T F

(iv) $\{(1,2,3),(4,5,6),(7,8,9)\}$ is a basis for $\mathbb{R}^3$. T F

Hint: $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \sim \begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix} \sim \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix}$

Q2/ Let $B = \{(1,2), (2,1)\}$,

$\underset{\sim}{u} = (1,0)$, $\underset{\sim}{v} = (0,1)$, $\underset{\sim}{w} = (1,1)$.

Row reduce

$$\left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 1 \\ 0 & -3 & -2 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2/3 & -1/3 & 1/3 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 0 & -1/3 & 2/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 & 1/3 \end{array}\right]$$

(a) Working over $\mathbb{R}$, find

$$[\underset{\sim}{u}]_B = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}, \quad [\underset{\sim}{v}]_B = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$$

$$[\underset{\sim}{w}]_B = [\underset{\sim}{u}]_B + [\underset{\sim}{v}]_B = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

(b) Working over $\mathbb{Z}_5$, find

$$[\underset{\sim}{u}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad [\underset{\sim}{v}]_B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$[\underset{\sim}{w}]_B = [\underset{\sim}{u}]_B + [\underset{\sim}{v}]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Q3/ Working over $\mathbb{Z}_3$, put

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}.$$

(a) Row reduce M:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Row reduce M^T:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) What is the rank of M? $\boxed{3}$

What is the nullity of M? $\boxed{1}$

(d) Find the null space $M^\perp$ of M:

$$M^\perp = \left\{ \begin{bmatrix} 2t \\ t \\ t \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{Z}_3 \right\} = \left\langle \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle.$$

(e) Find a basis for $M^\perp$: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Q4/ Working over $\mathbb{Z}_5$, put

$$M = \begin{bmatrix} 2 & 3 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 2 & 3 & 2 \end{bmatrix}.$$

(a) Row reduce M:

$$\begin{bmatrix} 2 & 3 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 2 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 4 & 4 & 4 & 4 \\ 0 & 1 & 2 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

(b) What is the rank of M? $\boxed{3}$

What is the nullity of M? $\boxed{2}$

(c) Find the null space $M^\perp$ of M:

$$\left\{ \begin{bmatrix} 3s+2t \\ s \\ 3s+4t \\ s \\ t \end{bmatrix} \,\middle|\, s, t \in \mathbb{Z}_5 \right\} = \left\langle \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

(d) Find a basis for $M^\perp$:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Q5/ Working over $\mathbb{Z}_5$, find the rank and nullity of

$$M = \begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 2 & 0 & 1 & 3 & 3 \end{bmatrix}$$

and a basis for $M^\perp$.

Solution:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 2 & 0 & 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 4 & 4 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

so $\operatorname{rank}(M) = 2$, $\operatorname{nullity}(M) = 3$.

$$M^\perp = \left\{ \begin{bmatrix} 2s+t+u \\ 4s+4t \\ s \\ t \\ u \end{bmatrix} \;\middle|\; s,t,u \in \mathbb{Z}_5 \right\}$$

with basis $\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Q6/ (a) Find the following cross-product:

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 17\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$$

and hence write down the equation of the plane through the origin spanned by

$$\underset{\sim}{v} = (1,2,3) \quad , \quad \underset{\sim}{w} = (2,-3,4) :$$

$$\boxed{17x + 2y - 7z = 0}$$

(b) Row reduce to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2/7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \end{bmatrix}$$

and use this to find the span of $\underset{\sim}{v}$ and $\underset{\sim}{w}$:

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \left\{ s(1,0,{}^{17}/_{7}) + t(0,1,{}^{2}/_{7}) \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \left\{ \left(s, t, \frac{17s+2t}{7}\right) \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \left\{ (x,y,z) \;\middle|\; z = \frac{17x+2y}{7} \right\} = \left\{ (x,y,z) \;\middle|\; 17x+2y-7z=0 \right\}$$

Q6/ (c) Use part (b) to deduce the equation of the plane P through the origin containing $(1,2,3)$ and $(2,-3,4)$:

$$\boxed{17x + 2y - 7z = 0}$$

(d) Row reduce the following:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 8 & -5 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \\ 0 & -21 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \\ 0 & -5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 3/7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) True or False:

(i) $(8,-5,18) \in P$ T F

(ii) $(3,1,8) \in P$ T F

Q7/ Row reduce the following:

$$\left[\begin{array}{cc|cc} 1 & 2 & 8 & 3 \\ 2 & -3 & -5 & 1 \\ 3 & 4 & 18 & 8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 8 & 3 \\ 0 & -7 & -21 & -5 \\ 0 & -2 & -6 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 8 & 3 \\ 0 & 1 & 3 & 5/7 \\ 0 & 1 & 3 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 11/7 \\ 0 & 1 & 3 & 5/7 \\ 0 & 0 & 0 & -3/14 \end{array}\right]$$

True or False:

(i) $(8,-5,18)$ is a linear combination of $(1,2,3)$ and $(2,-3,4)$. T F

(ii) $(3,1,8)$ is a linear combination of $(1,2,3)$ and $(2,-3,4)$. T F

For the true statement, find the linear combination:

$$(8,-5,18) = 2(1,2,3) + 3(2,-3,4)$$