MATH2022 Week 08

Worksheet

MATH 2022 Week 8 Worksheet

Q1/ Find all powers of 3 in $\mathbb{Z}_7$:

$3^1 = \boxed{}$, $3^2 = \boxed{}$, $3^3 = \boxed{}$,

$3^4 = \boxed{}$, $3^5 = \boxed{}$, $3^6 = \boxed{}$

Thus, under multiplication,

$\langle 3 \rangle =$

True or false:

$\mathbb{Z}_7 \setminus \{0\}$ is a cyclic group. T F

Find the rule for an isomorphism from $\mathbb{Z}_7 \setminus \{0\}$ under multiplication to $\mathbb{Z}_6$ under addition:

$3^i \mapsto \boxed{}$ for $i = 0, \ldots, 5$.

Q2/ Find all powers of 3 in $\mathbb{Z}_{13}$:

$3^1 = 3$, $3^2 =$

Thus

$$\langle 3 \rangle = \boxed{}$$

Find all powers of 2 in $\mathbb{Z}_{13}$:

$2^1 = 2$, $2^2 =$

Thus

$$\langle 2 \rangle = \boxed{}$$

<u>True or false:</u>

$\mathbb{Z}_{13} \setminus \{0\}$ is a cyclic group. T F

Find the rule for an isomorphism from $\mathbb{Z}_{13} \setminus \{0\}$ to $(\mathbb{Z}_{12}, +)$:

$$2^i \longmapsto \boxed{} \quad \text{for } i = 0, \dots, 11.$$

Q3/ The symbols $\langle x \rangle$ need to be read in context. In this exercise generation is with respect to **addition**.

True or false:

		T F
(a)	$\mathbb{Z}_6 = \langle 1 \rangle$	T F
(b)	$\mathbb{Z}_6 = \langle 2 \rangle$	T F
(c)	$\mathbb{Z}_6$ is cyclic.	T F
(d)	$\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,1) \rangle$	T F
(e)	$\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1,2) \rangle$	T F
(f)	$\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.	T F
(g)	$\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle (1,1) \rangle$	T F
(h)	$\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle (1,2) \rangle$	T F
(i)	$\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic	T F
(j)	$\mathbb{Z}_3 \times \mathbb{Z}_4$ is cyclic	T F
(k)	$\mathbb{Z}_4 \times \mathbb{Z}_6$ is cyclic	T F

Q4/ Let $G = \langle \alpha, \beta \rangle$ be the group of symmetries of a regular n-gon where $n \geq 3$ and

α = rotation $2\pi/n$,

β = reflection in an axis of symmetry.

Recall that $\alpha^n = \beta^2 = 1$ and

$\boxed{\alpha\beta = \beta\alpha^{-1}}$ and $\boxed{\beta\alpha = \alpha^{-1}\beta}$

Put $\boxed{\gamma = \alpha^4 \beta^3 \alpha^{-2} \beta \alpha^5 \beta \alpha^{-3}}$

Simplify γ in the following cases:

(a) $n = 5$:

(b) $n = 6$:

(c) $n = 7$:

Q5/ Consider the permutations

$$\alpha = (1\ 2\ 3)(4\ 5)(6\ 7\ 8),$$

$$\beta = (2\ 4\ 6)(3\ 7)(1\ 5\ 8).$$

Find

$\alpha\beta =$

$\beta\alpha =$

$\beta^{-1}\alpha\beta =$

$\alpha^{-1}\beta\alpha =$

Find a permutation δ such that

$$\beta = \delta^{-1}\alpha\delta.$$

Answer (one of many): $\delta =$

How many such δ are possible?

$\boxed{}$

Q6/ Consider the 15-puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Consider the following configuration

2	14	4	6
10	1	15	7
8	9	13	11
12	5	3	

Write down the associated permutation:

Is it even or odd?

Is the configuration possible?

Q7/ Consider the 15-puzzle:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Consider the following configuration:

1	5	8	10
11	2	6	9
14	12	3	7
	15	13	4

Write down the associated permutation where the blank square is labelled 16:

Is it even or odd?

Is the configuration possible?

Q8/ Working over $\mathbb{Z}_2$, complete the following multiplication table mod x^3+x+1, that is, $x^3+x+1=0$, so $\boxed{x^3 = x+1 = 1+x}$.

	0	1	x	$1+x$	x^2	$1+x^2$	$x+x^2$	$1+x+x^2$
0								
1								
x								
$1+x$								
x^2								
$1+x^2$								
$x+x^2$								
$1+x+x^2$								

Do we get a field? $\boxed{}$

The nonzero elements form a cyclic group, and each element $\neq 1$ is a generator. Verify this for powers of x: