THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2011
TOPICS IN DISCRETE MATHEMATICS

June 2004 Lecturer: David Easdown

Time allowed: two hours

Instructions to candidates

This examination paper comprises nine questions split into two sections.

There are two questions in **Section A**, each worth 10 marks.

There are seven questions in **Section B**, worth 5, 5, 5, 6, 6, 14 and 19 marks respectively.

The total number of marks available is 80, but full marks may be awarded for achieving 70 or more marks.

Both sections should be attempted.

Write your answers to **Section A** in the places indicated on the yellow question sheet. Write your name and student number on that sheet.

Write your answers to **Section B** in the answer booklet provided.

Place your completed answers for **Section A** (on the yellow sheet) inside the answer booklet, to be returned together to the examiner.

No notes or books are allowed. A University supplied calculator is permitted.

Section B

Write your answers to this section in the answer booklet.

B1. Suppose A and B are sets and $|A| = 12$, $|B| = 17$.

(a) Find $|A \cup B|$ given that $|A \cap B| = 11$.

(b) Find $|A \cap B|$ given that $|A \cup B| = 24$.

(c) Prove that it is impossible to find a set C such that

$$|C| = 14\,, \quad |A \cap B| = 9\,, \quad |A \cap B \cap C| = 5\,, \quad |A \cup B \cup C| = 30\,.$$

[$1 + 1 + 3 = 5$ marks]

B2. Consider the alphabet $\Sigma = \{a, b, c, d, e, f\}$. Evaluate the answers to each of the following questions as integers. For example, $\binom{7}{2}$ evaluates to the integer 21.

(a) How many *ordered sequences without repetition* of 3 letters can be chosen from Σ?

(b) How many *ordered sequences with repetition* of 3 letters can be chosen from Σ?

(c) In how many ways, *unordered without repetition*, can you choose 3 letters from Σ?

(d) In how many ways, *unordered with repetition*, can you choose 3 letters from Σ?

[$1 + 1 + 1 + 2 = 5$ marks]

B3. Find simple expressions for the generating functions of the following sequences. For example, the sequence $1, 1, 1, 1, \ldots$ has generating function $G(z) = \dfrac{1}{1-z}$.

(a) $1, -1, 1, -1, \ldots$

(b) $1, 4, 4^2, 4^3, \ldots$

(c) $0, 0, 1, 1, 1, 1, \ldots$

(d) $1, -2, 3, -4, \ldots$

[$1 + 1 + 1 + 2 = 5$ marks]

B4. If today is Sunday, what day will it be after 50^{50} days have elapsed? If today is Sunday and it is 12 noon, what day and time will it be after 50^{50} hours have elapsed? Show your working.

[$2 + 4 = 6$ marks]

B5. Solve the following system of equations for the least positive integer x:

$$\begin{aligned} x &= 0 \pmod{2} \\ x &= 1 \pmod{11} \\ x &= 23 \pmod{25} \end{aligned}$$

[6 marks]

B6. Recall that if $f(N)$ and $g(N)$ are nonnegative functions of a natural number N then $f(N) = \Omega(g(N))$ if there exist positive constants K and N_0 such that $f(N) \geq Kg(N)$ for all $N \geq N_0$.

(a) Define what is meant by $f(N) = O(g(N))$, and verify that this implies $g(N) = \Omega(f(N))$.

(b) Show that $N \log N + 100 - 2N = \Omega(N \log N)$ (where you may assume all logarithms are natural).

Suppose that $\mathcal{C}$ is a convex hull algorithm which inputs a finite set of points in the plane and outputs vertices of the boundary of the convex hull as a polygon in standard form. Let $\mathcal{S}$ be the following algorithm:

Algorithm $\mathcal{S}$: Input distinct positive reals $x_1, x_2, \ldots, x_N$.

Step (1): Form $X = \{(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)\}$.

Step (2): Input X to $\mathcal{C}$ which will output vertices

$$(y_1, y_1^2)\,,\ (y_2, y_2^2)\,,\ \ldots\,,\ (y_N, y_N^2)\,.$$

Step (3): Halt and output the sequence $y_1, y_2, \ldots, y_N$.

(c) What does the algorithm $\mathcal{S}$ achieve?

(d) Let the running time of $\mathcal{S}$ be $T_{\mathcal{S}}(N)$ and of $\mathcal{C}$ be $T_{\mathcal{C}}(N)$. Explain briefly why $T_{\mathcal{S}}(N) = T_{\mathcal{C}}(N) + O(N)$.

(e) It is an important theorem in algorithm analysis that all sorting algorithms have running time which is $\Omega(N \log N)$. Use this theorem to deduce that $T_{\mathcal{C}}(N) = \Omega(N \log N)$.

[$2 + 3 + 1 + 2 + 6 = 14$ marks]

B7. Recall that the Fibonacci numbers $a_0, a_1, \ldots, a_n, \ldots$ are defined by $a_0 = a_1 = 1$ and the recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

(a) Write down the characteristic equation of this recurrence and find its roots. Hence derive the formula

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

(b) Prove by induction that the greatest common divisor of consecutive Fibonacci numbers is 1.

(c) Describe briefly the outcome of applying the Euclidean Algorithm to consecutive Fibonacci numbers a_n and a_{n-1} for $n \geq 1$. How many equations will be displayed? Show that if $N = a_n$ then this number of equations is $\Omega(\log N)$.

[$6 + 6 + 7 = 19$ marks]

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