THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2069
Discrete Mathematics and Graph Theory
PAPER 1

June 2005 Lecturer: David Easdown

Time allowed: one and a half hours

Instructions to candidates

This paper examines Discrete Mathematics and comprises seven questions split into two sections.

There are two questions in **Section A**, each worth 10 marks.

There are five questions in **Section B**, each worth 10 marks.

The total number of marks available is 70.

Both sections should be attempted.

Write your answers to **Section A** in the places indicated on the yellow question sheet. Write your name and student number on that sheet.

Write your answers to **Section B** in the answer booklet provided.

Place your completed answers for **Section A** (on the yellow sheet) inside the answer booklet, to be returned together to the examiner.

No notes or books are allowed. A University supplied calculator is permitted.

Section B

Write your answers to this section in the answer booklet.

B1.

(a) Prove by induction that, for every positive integer n,

$$1^2 + 2^2 + \ldots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6} \; .$$

(b) Define the floor and ceiling functions and verify that if N is a positive integer then

$$N \;=\; \lfloor N/2 \rfloor + \lceil N/2 \rceil \; .$$

[$6 + 4 = 10$ marks]

B2.

(a) Find a simple expression for the generating function of the sequence

$$1\,,\; 2\,,\; 2^2\,,\; 2^3\,,\; 2^4\,,\; 2^5\,,\; \ldots$$

(b) Write down the sequence associated with the generating function

$$G(z) \;=\; \frac{z}{(1-z)^2} \; .$$

(c) Let $G(z)$ be the generating function for the sequence defined by $a_0 = 4$ and $a_n = 2a_{n-1} - 1$ for $n \geq 1$. Verify that

$$G(z)(1-2z) = \frac{4-5z}{1-z}$$

and use partial fractions to derive the formula $a_n = 3(2^n) + 1$.

[$1 + 3 + 6 = 10$ marks]

B3.

(a) Given that the time now is 4 pm on Wednesday, what time and day will it be after 47^{74} hours have elapsed? (Hint: first work modulo 24 and then modulo 7.)

(b) Use the Euclidean Algorithm (forwards then backwards) to find 34^{-1} in $\mathbb{Z}_{57}$, and then solve the following for x:

$$34x \;=\; 25 \pmod{57} \; .$$

[$6 + 4 = 10$ marks]

B4. Suppose we have a strictly increasing sequence $\mathcal{S}$ of $N = 2^k$ real numbers: $a_1 < a_2 < \ldots < a_{N-1} < a_N$. Put

$$\mathcal{S}_L = \{a_1, \ldots, a_{N/2}\} \qquad \text{and} \qquad \mathcal{S}_R = \{a_{N/2+1}, \ldots, a_N\} ,$$

and write

$$M_L = M_L(\mathcal{S}) = a_{N/2} , \quad M_R = M_R(\mathcal{S}) = a_{N/2+1} .$$

Consider the following recursive algorithm $\mathrm{B}(\mathcal{S})$ associated with $\mathcal{S}$:

$\mathrm{B}(\mathcal{S})$: Input a real number λ.

(1) If $\mathcal{S} = \{X\}$ output $(-\infty, X)$ if $\lambda < X$; output $[X, \infty)$ otherwise, and stop.

(2) If $M_L \leq \lambda < M_R$ output $[M_L, M_R)$ and stop.

(3) If $\lambda < M_L$ then input λ to $\mathrm{B}(\mathcal{S}_L)$.

(4) Otherwise input λ to $\mathrm{B}(\mathcal{S}_R)$.

(a) Describe the output when $\mathrm{B}\big(1, 2, \ldots, 64\big)$ is applied to each of the inputs

$$\lambda = 32.97 , \qquad \lambda = 12.001 \qquad \text{and} \qquad \lambda = -8 .$$

You are not being asked to describe the steps taken by the algorithm.

(b) Explain why the running time $T(N)$ of the algorithm, as a function of the number $N(= 2^k)$ of elements in the original ordered list, is described by

$$T(N) = T(N/2) + O(1) .$$

(c) Solve the recurrence of (b) to show that $T(N) = O(\log N)$.

[3 + 3 + 4 = 10 marks]

B5.

(a) State the converse and the contrapositive of the following implications:

(i) If the sun is shining then I am putting on my sunglasses.

(ii) It is necessary to have a student number to sit this exam.

(b) Express the following statements about integers using quantifiers and usual mathematical symbols:

(i) The product of two negative integers is positive.

(ii) The set $\mathbb{Z}$ has no largest or smallest element.

(c) We will invent a logical connective called $\triangle$. The truth table for $\triangle$ is

P	Q	$P \triangle Q$
T	T	T
T	F	F
F	T	X
F	F	Y

for some truth values X and Y . We stipulate further that

$$Q \triangle P \qquad \text{and} \qquad \sim Q \triangle \sim P$$

are logically equivalent. Prove that $X = F$ and $Y = T$. How does $\triangle$ relate to usual implication?

[$3 + 3 + 4 = 10$ marks]

THIS IS THE LAST PAGE OF THE EXAMINATION PAPER