THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH2069
Discrete Mathematics and Graph Theory
PAPER 1

June 2005 Lecturer: David Easdown

Time allowed: one and a half hours

Instructions to candidates

This paper examines Discrete Mathematics and comprises seven questions split into two sections.

There are two questions in Section A, each worth 10 marks.

There are five questions in Section B, each worth 10 marks.

The total number of marks available is 70.

Both sections should be attempted.

Write your answers to Section A in the places indicated on the yellow question sheet. Write your name and student number on that sheet.

Write your answers to Section B in the answer booklet provided.

Place your completed answers for Section A (on the yellow sheet) inside the answer booklet, to be returned together to the examiner.

No notes or books are allowed. A University supplied calculator is permitted.
Section B

Write your answers to this section in the answer booklet.

B1.
(a) Prove by induction that, for every positive integer \( n \),
\[
1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]
(b) Define the floor and ceiling functions and verify that if \( N \) is a positive integer then
\[
N = \lfloor N/2 \rfloor + \lceil N/2 \rceil.
\]

[6 + 4 = 10 marks]

B2.
(a) Find a simple expression for the generating function of the sequence
\[
1, 2, 2^2, 2^3, 2^4, 2^5, \ldots
\]
(b) Write down the sequence associated with the generating function
\[
G(z) = \frac{z}{(1-z)^2}.
\]
(c) Let \( G(z) \) be the generating function for the sequence defined by \( a_0 = 4 \) and \( a_n = 2a_{n-1} - 1 \) for \( n \geq 1 \). Verify that
\[
G(z)(1-2z) = \frac{4 - 5z}{1 - z}
\]
and use partial fractions to derive the formula \( a_n = 3(2^n) + 1 \).

[1 + 3 + 6 = 10 marks]

B3.
(a) Given that the time now is 4 pm on Wednesday, what time and day will it be after \( 47^{74} \) hours have elapsed? (Hint: first work modulo 24 and then modulo 7.)

(b) Use the Euclidean Algorithm (forwards then backwards) to find \( 34^{-1} \) in \( \mathbb{Z}_{57} \), and then solve the following for \( x \):
\[
34x = 25 \pmod{57}.
\]

[6 + 4 = 10 marks]
**B4.** Suppose we have a strictly increasing sequence \( S \) of \( N = 2^k \) real numbers: \( a_1 < a_2 < \ldots < a_{N-1} < a_N \). Put

\[
S_L = \{a_1, \ldots, a_{N/2}\} \quad \text{and} \quad S_R = \{a_{N/2+1}, \ldots, a_N\},
\]

and write

\[
M_L = M_L(S) = a_{N/2}, \quad M_R = M_R(S) = a_{N/2+1}.
\]

Consider the following recursive algorithm \( B(S) \) associated with \( S \):

\begin{enumerate}
  \item B(S): Input a real number \( \lambda \).
  \item If \( S = \{X\} \) output \((-\infty, X)\) if \( \lambda < X \); output \([X, \infty)\) otherwise, and stop.
  \item If \( M_L \leq \lambda < M_R \) output \([M_L, M_R)\) and stop.
  \item If \( \lambda < M_L \) then input \( \lambda \) to \( B(S_L) \).
  \item Otherwise input \( \lambda \) to \( B(S_R) \).
\end{enumerate}

(a) Describe the output when \( B(1, 2, \ldots, 64) \) is applied to each of the inputs \( \lambda = 32.97 \), \( \lambda = 12.001 \) and \( \lambda = -8 \).

You are not being asked to describe the steps taken by the algorithm.

(b) Explain why the running time \( T(N) \) of the algorithm, as a function of the number \( N(= 2^k) \) of elements in the original ordered list, is described by

\[
T(N) = T(N/2) + O(1).
\]

(c) Solve the recurrence of (b) to show that \( T(N) = O(\log N) \).

\[3 + 3 + 4 = 10 \text{ marks}]
B5.

(a) State the converse and the contrapositive of the following implications:

(i) If the sun is shining then I am putting on my sunglasses.

(ii) It is necessary to have a student number to sit this exam.

(b) Express the following statements about integers using quantifiers and usual mathematical symbols:

(i) The product of two negative integers is positive.

(ii) The set \( Z \) has no largest or smallest element.

(c) We will invent a logical connective called \( \triangle \). The truth table for \( \triangle \) is

\[
\begin{array}{ccc}
P & Q & P \triangle Q \\
T & T & T \\
T & F & F \\
F & T & X \\
F & F & Y \\
\end{array}
\]

for some truth values \( X \) and \( Y \). We stipulate further that

\[ Q \triangle P \quad \text{and} \quad \sim Q \triangle \sim P \]

are logically equivalent. Prove that \( X = F \) and \( Y = T \). How does \( \triangle \) relate to usual implication?

\[3 + 3 + 4 = 10 \text{ marks}\]