The University of Sydney

MATH2969 Discrete Mathematics and Graph Theory (Advanced)

Semester 1 **Discrete Maths: Assignment Solutions** 2006

1. The store can fulfil an order for 44 pencils by supplying 4 packs of 5 and 2 packs of 12, which starts an induction. Suppose $k \geq 44$ and the store can fulfil the order for k pencils using A packs of 5 and B packs of 12, so $5A + 12B = k$. If $B \geq 2$ then the order for

$$k + 1 \;=\; 5A + 12B + 25 - 24 \;=\; 5(A + 5) + 12(B - 2)$$

pencils is fulfilled using $(A + 5)$ packs of 5 and $(B - 2)$ packs of 12. If $B \leq 1$ then

$$5A \;=\; k - 12B \;\geq 44 - 12 \;=\; 32$$

so that $A \geq 7$, in which case the order for

$$k + 1 \;=\; 5A + 12B + 36 - 35 \;=\; 5(A - 7) + 12(B + 3)$$

pencils is fulfilled using $(A - 7)$ packs of 5 and $(B + 3)$ packs of 12. In either case the order for $k + 1$ pencils is fulfilled, establishing the inductive step and completing the proof.

The order of 43 pencils can't be filled because all multiples of 12 less than 43, namely 0, 12, 24 and 36, do not differ from 43 by a multiple of 5.

2. The Fibonacci number $b_n = 0 \pmod 2$ iff $n = 2 \pmod 3$. Here is a proof:

The induction begins because $b_0 = b_1 = 1 \neq 0 \pmod 2$ and $0, 1 \neq 2 \pmod 3$. Suppose that $k \geq 3$ and the claim holds for $n = k - 1, k - 2$.

If $k = 2 \pmod 3$ then $k - 1 = 1, k - 2 = 0 \pmod 3$, so $b_k = b_{k-1} + b_{k-2} = 1 + 1 = 0 \pmod 2$.

If $k = 1 \pmod 3$ then $k - 1 = 0, k - 1 = 2 \pmod 3$, so $b_k = b_{k-1} + b_{k-2} = 1 + 0 = 1 \pmod 2$.

If $k = 0 \pmod 3$ then $k - 1 = 2, k - 2 = 1 \pmod 3$, so $b_k = b_{k-1} + b_{k-2} = 0 + 1 = 1 \pmod 2$.

This completes the inductive step, and the claim follows by induction.

3. Put $G(z) = \sum_{n=0}^{\infty} a_n z^n$ so

$$
\begin{aligned}
G(z) &= a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n \;=\; a_0 + a_1 z + \sum_{n=2}^{\infty} (r a_{n-1} + s a_{n-2}) z^n \\
&= a_0 + a_1 z + r \sum_{n=2}^{\infty} a_{n-1} z^n + s \sum_{n=2}^{\infty} a_{n-2} z^n \\
&= a_0 + a_1 z + rz \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + s z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} \\
&= a_0 + a_1 z + rz \sum_{n=1}^{\infty} a_n z^n + s z^2 \sum_{n=0}^{\infty} a_n z^n \\
&= a_0 + a_1 z + rz(G(z) - a_0) + s z^2 G(z)\ ,
\end{aligned}
$$

giving

$$G(z)(1 - rz - sz^2) \;=\; a_0 + a_1 z - a_0 r z$$

so that

$$G(z) \;=\; \frac{a_0 + (a_1 - a_0 r)z}{1 - rz - sz^2} \;=\; \frac{a_0 + (a_1 - a_0 r)z}{(1 - \lambda_1 z)(1 - \lambda_2 z)}$$

since $1 - rz - sz^2 = (z^{-1} - \lambda_1)(z^{-1} - \lambda_2)z^2 = (1 - \lambda_1 z)(1 - \lambda_2 z)$.

If $\lambda_1 \neq \lambda_2$ then we get the partial fraction decomposition

$$G(z) \;=\; \frac{A}{1 - \lambda_1 z} + \frac{B}{1 - \lambda_2 z}$$

for some constants A and B, so that, using the geometric series, we get

$$G(z) \;=\; A \sum_{n=0}^{\infty} \lambda_1^n z^n + B \sum_{n=0}^{\infty} \lambda_2^n z^n \;=\; \sum_{n=0}^{\infty} (A\lambda_1^n + B\lambda_2^n) z^n$$

so that $a_n = C_1 \lambda_1^n + C_2 \lambda_2^n$ for each n where $C_1 = A$ and $C_2 = B$.

If $\lambda_1 = \lambda_2$ then we get the partial fraction decomposition

$$G(z) \;=\; \frac{A}{1 - \lambda_1 z} + \frac{B}{(1 - \lambda_1 z)^2}$$

for some constants A and B, so that, using the geometric series and its derivative,

$$G(z) \;=\; A \sum_{n=0}^{\infty} \lambda_1^n z^n + B \sum_{n=0}^{\infty} (n+1) \lambda_1^n z^n \;=\; \sum_{n=0}^{\infty} \big((A+B)\lambda_1^n + Bn\lambda_1^n\big) z^n$$

so that $a_n = C_1 \lambda_1^n + C_2 n \lambda_1^n$ for each n where $C_1 = A + B$ and $C_2 = B$.

4. Note that $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ and differentiating twice yields

$$\frac{1}{(1-z)^3} \;=\; \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n \; .$$

Hence

$$\begin{aligned}
\frac{z(1+z)}{(1-z)^3} \;&=\; z(1+z) \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n \\
&=\; \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^{n+1} + \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^{n+2} \\
&=\; \sum_{m=1}^{\infty} \frac{(m+1)m}{2} z^m + \sum_{m=2}^{\infty} \frac{m(m-1)}{2} z^m \\
&=\; \sum_{m=0}^{\infty} \frac{(m+1)m}{2} z^m + \sum_{m=0}^{\infty} \frac{m(m-1)}{2} z^m \\
&=\; \sum_{m=0}^{\infty} \frac{(m+1)m + m(m-1)}{2} z^m \;=\; \sum_{m=0}^{\infty} m^2 z^m \;=\; \sum_{n=0}^{\infty} n^2 z^n \; .
\end{aligned}$$

5. Observe first that

$$N! \;=\; N \times (N-1) \times (N-2) \times \ldots \times 2 \times 1 \;\leq\; N^N$$

so

$$\log\, N! \;\leq\; \log N^N \;=\; N \log N \;=\; O(N \log N) \; .$$

Now, for $N \geq 2$,

$$\frac{N}{3} \;\leq\; \left\lfloor \frac{N}{2} \right\rfloor \;\leq\; \left\lceil \frac{N}{2} \right\rceil$$

so

$$\begin{aligned}
\left(\frac{N}{3}\right)^{N/3} \;&\leq\; \left\lfloor \frac{N}{2} \right\rfloor^{\lceil N/2 \rceil} \\
&\leq\; N \times (N-1) \times (N-2) \times \ldots \times \left\lfloor \frac{N}{2} \right\rfloor \times \ldots \times 2 \times 1 \\
&=\; N!
\end{aligned}$$

so that

$$\frac{N}{3} \log N - \frac{N}{3} \log 3 \;=\; \frac{N}{3} \log \frac{N}{3} \;\leq\; \log\, N!$$

giving

$$\frac{N}{3} \log N \;\leq\; \log\, N! + \frac{N}{3} \log 3$$

whence

$$\begin{aligned} N \log N &= O(\frac{N}{3} \log N) &= O(\log N! + \frac{\log 3}{3} N) \\ &= O(\log N! + N) &= O(\log N!) \ . \end{aligned}$$

6. Let the nth element of the sequence be k, so

$$1 + 2 + \ldots + (k-1) + 1 \ \leq \ n \ \leq \ 1 + 2 + \ldots + k$$

giving

$$\frac{(k-1)k}{2} + 1 \ \leq \ n \ \leq \ \frac{k(k+1)}{2}$$

whence

$$k^2 - k + 2 \ \leq \ 2n \ \leq \ k^2 + k \ .$$

Thus

$$\left(k - \frac{1}{2}\right)^2 + \frac{7}{4} \ \leq \ 2n \ < \ \left(k + \frac{1}{2}\right)^2$$

so

$$k - \frac{1}{2} \ \leq \ \sqrt{2n - \frac{7}{4}} \ < \ \sqrt{2n} \ < \ k + \frac{1}{2} \ ,$$

whence

$$k \ < \ \sqrt{2n} + \frac{1}{2} \ < \ k + 1 \ .$$

By definition of the floor function, this proves

$$k \ = \ \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor \ .$$

7. Using the Euclidean algorithm we know there are integers z and w such that

$$mz + nw \ = \ 1 \ .$$

If $z > 0$ then we are done (taking $x = z$, $y = w$). Suppose now $z < 0$ and let x be any positive number differing from z by a multiple of n, say $x = z + kn$ for a sufficiently large positive integer k. Put $y = w - km$ .. Then

$$\begin{aligned} mx + ny &= m(z + kn) + n(w - km) \\ &= mz + mkn + nw - nkm \\ &= mz + nw \ = \ 1 \ . \end{aligned}$$

8. (a) The encoded message is

0261|2909|0666|2401|1598|3039

(b) $3,763 = 53 \times 71$

(c) This is straightforward and is a good check on whether you have the method right in order to tackle part (d).

(d) The message is

YOU_HAVE_FOUND_A_WEAPON_OF_MATHS_DECONSTRUCTION_