THE UNIVERSITY OF SYDNEY

MATH2969 DISCRETE MATHEMATICS AND GRAPH THEORY (ADVANCED)

Semester 1 **Discrete Mathematics: Assignment** 2006

This assignment is to be handed in by advanced students only, and is due Friday 5 May 2006 (end of Week 7). **Only answers to starred questions *1, *3, *6, *7 are to be handed in.** *Whilst general discussion with other students in the course is to be encouraged, the substance of this assignment and all written detail must be your own work. Acknowledge any sources or assistance. The remaining questions should be attempted and may be related to exam questions. Solutions will become available from 29 May for students to selfmark answers to unstarred questions.*

***1.** A store sells pencils in packs of 5 or 12. You need n pencils. Prove that the store can fill your exact order provided $n \geq 44$. What if $n = 43$?

2. Which terms of the Fibonacci sequence are even? Prove your answer by induction.

***3.** Consider the following homogeneous recurrence relation

$$a_n - ra_{n-1} - sa_{n-2} \;=\; 0$$

where r and s are real constants and $n \geq 2$. The characteristic equation is

$$\lambda^2 - r\lambda - s \;=\; 0$$

which has (complex) roots λ_1 and λ_2 say. Use generating functions and partial fractions to prove that the recurrence relation has the general solution

$$a_n \;=\; \begin{cases} C_1\lambda_1^n + C_2\lambda_2^n & \text{if } \lambda_1 \neq \lambda_2 \\ C_1\lambda_1^n + C_2 n\lambda_1^n & \text{if } \lambda_1 = \lambda_2 \end{cases}$$

for some constants C_1 and C_2 and all $n \geq 0$.

4. Verify that $\dfrac{z(1+z)}{(1-z)^3} \;=\; \displaystyle\sum_{n=0}^{\infty} n^2 z^n$.

5. Verify that the functions $\log N!$ and $N \log N$ grow at the same rate, in the sense of being Big-Oh of each other.

***6.** Consider the sequence

$$1\,,\ 2\,,\ 2\,,\ 3\,,\ 3\,,\ 3\,,\ 4\,,\ 4\,,\ 4\,,\ 4\,,\ 5\,,\ 5\,,\ 5\,,\ 5\,,\ 5\,,\ \ldots$$

Prove that the nth element is

$$\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor .$$

*7. Suppose m and n are coprime positive integers and observe that

$$mn + (-n)m \;=\; 0\,.$$

Exploit this observation to show that there exist integers x, y with x positive such that

$$mx + ny \;=\; 1\,.$$

(The Euclidean algorithm finds integers x and y but does not guarantee that x is positive.)

8. Associate with each letter of the alphabet $A, B, \ldots, Y, Z$ the two digits

$$01\,,\; 02\,,\; \ldots\,,\; 25\,,\; 26$$

respectively. Associate with a space the two digits 00. Thus

THESE WORDS

plus an extra space becomes

20080519050023151804 1900

Divide this into groups of 4 digits:

2008|0519|0500|2315|1804|1900

Now encode each 4 digit number N by forming the remainder of N^3 after division by $3,763$.

(a) What is the encoded message?

(b) Factorize $3,763$ as a product of primes p and q.

(c) Use the method derived from Fermat's Little Theorem and the Chinese Remainder Theorem to check that you decode back to the original strings of digits.

(d) Now use the method to decode the following message which you have just intercepted:

1369|1116|1965|0703|0216|0091|2365|0001|0878|3330|3753|1585|
2168|2197|0783|0415|0064|2630|3150|1092|1983|3559|0624|1585

The first student or group of students enrolled in MATH2969 to email David (de@maths.usyd.edu.au) the decoded message wins a chocolate orange!! A condition of entry is that the answer is found without consulting a person in another course and without using a computer or a programmable calculator. (An ordinary hand-held calculator such as used in exams is permissable).