THE UNIVERSITY OF SYDNEY

MATH2069/2969 DISCRETE MATHEMATICS AND GRAPH THEORY

Semester 1 **Discrete Mathematics Tutorial 1 (Week 2)** 2006

Starred questions are suitable for students enrolled in MATH2969 or for students aiming for a credit or higher.

1. Suppose T_n for $n \geq 0$ is defined recursively by
$$T_0 = 5\ , \qquad T_n = 2T_{n-1} + 1 \quad \text{for } n \geq 1\ .$$

(a) Find T_1, T_2, T_3, T_4 and T_5.

(b) Verify by induction that
$$T_n = 6(2^n) - 1$$
for all $n \geq 0$.

*(c) More generally, let k, c and d be constants where $c \neq 0, 1$. Verify that if
$$T_0 = k\ , \qquad T_n = cT_{n-1} + d \quad \text{for } n \geq 1$$
then
$$T_n = c^n \left(k - \frac{d}{1-c}\right) + \frac{d}{1-c}\ .$$
(In the Tower of Hanoi Problem $k = 0$, $c = 2$ and $d = 1$.) What happens if $c = 1$?

2. (a) Verify by induction or otherwise the following formula for sums of even integers:
$$2 + 4 + 6 + \ldots + 2n \ = \ n^2 + n\ .$$

*(b) More generally verify the following formula for the sum of a finite arithmetic sequence:
$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) \ = \ \frac{n}{2}[2a + (n - 1)d]\ .$$

3. (a) Verify by induction or otherwise the following formula for sums of powers of 2:
$$1 + 2 + 2^2 + \ldots + 2^n \ = \ 2^{n+1} - 1\ .$$

*(b) More generally verify the following formula for the sum of a finite geometric sequence when $r \neq 1$:
$$a + ar + ar^2 + \ldots + ar^n \ = \ \frac{a(1 - r^{n+1})}{1 - r}$$
and deduce the following well-known formula for an infinite geometric series when $|r| < 1$:
$$\frac{1}{1 - r} \ = \ 1 + r + r^2 + r^3 + \ldots$$

4. Verify by induction that $2^n < n!$ for all $n \geq 4$.

5. Prove that, for all $n \geq 1$,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} .$$

6. Verify that, for all integers $n \geq 1$,

(a) $n^3 + 5n$ is a multiple of 3.

*(b) $5^n - 4n - 1$ is divisible by 16.

7. Verify that

(a) $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

*(b) $(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3$.

**8. A store sells pencils in packs of 5 or 12. You need n pencils. Prove that the store can fill your exact order provided $n \geq 44$. What if $n = 43$?

**9. Find the error in the following argument that claims to prove that all the people in this room have the same height:

> We argue by induction on the number n of people in this room. If $n = 1$ then the claim is obviously true, which begins the induction. Suppose $k \geq 1$ and the claim holds for rooms with k people. Suppose $P_1, P_2, \ldots, P_{k+1}$ are the $k+1$ people in this room. By the inductive hypothesis $P_1, P_2, \ldots, P_k$ all have the same height, and also $P_2, \ldots, P_k, P_{k+1}$ all have the same height (since these two groups consisting of k people could each be separated out into a single room). But P_2 is in common, so all of $P_1, P_2, \ldots, P_k, P_{k+1}$ have the same height. This establishes the inductive step, and so the claim holds for all n by induction.

**10. Prove that a $2^n \times 2^n$ grid with exactly one subsquare missing can be tiled by 2×2 squares with one subsquare missing (L-shapes).

11. Solve the following recurrences:

(a) $a_n = 7a_{n-1}$ for $n \geq 1$ where $a_0 = 3$.

(b) $a_{n+1} = 5a_n$ for $n \geq 0$ where $a_0 = 2$.

(c) $a_n = -a_{n-1}$ for $n \geq 1$ where $a_0 = 4$.

12. Use the characteristic equation to solve the following recurrences:

(a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ where $a_0 = 2$, $a_1 = 5$.

(b) $a_n = 4a_{n-1} - 3a_{n-2}$ for $n \geq 2$ where $a_0 = -1$, $a_1 = 2$.

*(c) $a_n = -a_{n-2}$ for $n \geq 2$ where $a_0 = 4$, $a_1 = 6$ (as a formula in $i = \sqrt{-1}$).

13. Use the characteristic equation to solve the following recurrences:

(a) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ where $a_0 = 1$, $a_1 = 4$.

(b) $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$ where $a_0 = 2$, $a_1 = -3$.

**(c) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ for $n \geq 3$ where $a_0 = 2$, $a_1 = 4$, $a_2 = 16$.

14. A person opens a bank account with a deposit of $100 and adds $10 each month. The account earns interest at 6% per annum compounded monthly.

(a) Let a_n = amount in account after n months. Write down the initial condition and a recurrence for a_{n+1} in terms of a_n.

*(b) Find a formula for a_n in terms of n and check your answer by induction.

*(c) How much will be in the account after 10 years? 20 years? How long will the person take to become a millionaire?

15. A fly lands on a piece of food depositing a malignant single-celled organism which divides into two every 5 minutes.

(a) How many organisms will inhabit the food after one hour? How long will it take before there are one million organisms? 100 million organisms?

*(b) Suppose in addition that a natural antibiotic is activated which, after one hour has elapsed, kills 2,000 organisms initially and then each 5 minutes, so we may assume

$$a_{11} = 2^{11}, \qquad a_{n+1} = 2a_n - 2,000 \quad \text{for } n \geq 11,$$

where a_n denotes the number of organisms $5n$ minutes from the moment the fly landed. The food is safe to eat provided there are less than one million organisms. Find a general formula for a_n and the period of time for which the food remains safe.

16. A sequence is defined recursively by $a_0 = a_1 = 2$ and $a_n = a_{n-1} \cdot a_{n-2}$ for $n \geq 2$.

(a) Find a_2, a_3, a_4, a_5 and a_6.

*(b) Verify by induction that $a_n = 2^{b_n}$ where $b_0, b_1, b_2, \ldots$ is the Fibonacci sequence.

****17.** Which terms of the Fibonacci sequence are even? Prove your answer by induction.

18. Let a_n be the number of ways of forming a line of n people distinguished only by whether they are male (M) or female (F). For example there are four possibilities with 2 people:

$$FF\,, \quad FM\,, \quad MF\,, \quad MM\,.$$

Write down a recurrence for a_n and an explicit formula in terms of n.

***19.** Let b_n be the number of ways of forming a line of n people distinguished only by whether they are male (M) or female (F), such that no two males are next to each other. For example there are five possibilities with 3 people:

$$FFF\,, \quad FFM\,, \quad FMF\,, \quad MFF\,, \quad MFM\,.$$

Write down a recurrence for b_n. Do you recognize the sequence?

****20.** Companies A and B control the market for a certain product, each starting off with a 50% share. From one year to the next A retains 70% and loses to B the remaining 30% of its custom, whilst B retains 60% and loses to A the remaining 40% of its custom. Let A_n and B_n denote the market shares of A and B respectively after n years.

(a) Verify by induction that, for all $n \geq 0$,

$$A_n \;=\; -\frac{1}{14}\left(\frac{3}{10}\right)^n + \frac{4}{7}\,, \qquad B_n \;=\; \frac{1}{14}\left(\frac{3}{10}\right)^n + \frac{3}{7}\,.$$

(b) What happens to the market share in the long term?

(c) Derive the following homogeneous recurrence relation

$$A_{n+2} - \frac{13}{10}A_{n+1} + \frac{3}{10}A_n \;=\; 0\,,$$

find the roots of its characteristic equation, and rediscover the formulae in part (a).

(d) Prove that the long term market behaviour is independent of the original market shares of the two companies.