

1. (a) $T_1 = 11, T_2 = 23, T_3 = 47, T_4 = 95$ and $T_5 = 191$.

(b) The induction begins because $T_0 = 5 = 6(2^0) - 1$. Suppose $k \geq 1$ and $T_k = 6(2^k) - 1$. Then

$$T_{k+1} = 2(6(2^k) - 1) + 1 = 6(2)^{k+1} - 1,$$

establishing the inductive step and completing the proof.

* (c) The induction begins because

$$T_0 = k = c^0 \left(k - \frac{d}{1-c} \right) + \frac{d}{1-c}.$$

Suppose $\ell \geq 1$ and

$$T_\ell = c^\ell \left(k - \frac{d}{1-c} \right) + \frac{d}{1-c}.$$

Then

$$\begin{aligned} T_{\ell+1} &= cT_\ell + d = c \left[c^\ell \left(k - \frac{d}{1-c} \right) + \frac{d}{1-c} \right] + d \\ &= c^{\ell+1} \left(k - \frac{d}{1-c} \right) + \frac{cd}{1-c} + \frac{d-dc}{1-c} \\ &= c^{\ell+1} \left(k - \frac{d}{1-c} \right) + \frac{d}{1-c}, \end{aligned}$$

establishing the inductive step and completing the proof. If $c = 1$ then the recurrence becomes $T_n = T_{n-1} + d$ leading to the sequence

$$k, k + d, k + 2d, \dots, k + nd, \dots$$

with general term $T_n = k + nd$.

2. (a) The induction begins because $2 = 1^2 + 1$. Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$\begin{aligned} 2 + 4 + 6 + \dots + 2k + 2(k+1) &= k^2 + k + 2(k+1) \\ &= (k+1)^2 + k + 1, \end{aligned}$$

establishing the inductive step and completing the proof.

*(b) The induction begins because

$$a = \frac{1}{2}[2a + (1 - 1)d].$$

Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$\begin{aligned} a + (a + d) + (a + 2d) + \dots + (a + (k - 1)d) + (a + kd) \\ &= \frac{k}{2}[2a + (k - 1)d] + a + kd \\ &= ka + \frac{k}{2}(k - 1)d + a + kd \\ &= (k + 1)a + \frac{k}{2}(k + 1)d \\ &= \frac{k + 1}{2}(2a + kd), \end{aligned}$$

establishing the inductive step and completing the proof.

3. (a) The induction begins because $1 = 2^{0+1} - 1$. Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1,$$

establishing the inductive step and completing the proof.

*(b) The induction begins because

$$a = \frac{a(1 - r^{0+1})}{1 - r}.$$

Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$\begin{aligned} a + ar + ar^2 + \dots + ar^k + ar^{k+1} &= \frac{a(1 - r^{k+1})}{1 - r} + ar^{k+1} \\ &= \frac{a - ar^{k+1} + ar^{k+1} - ar^{k+2}}{1 - r} \\ &= \frac{a(1 - r^{k+2})}{1 - r}, \end{aligned}$$

establishing the inductive step and completing the proof. Thus if $|r| < 1$ then

$$\begin{aligned} 1 + r + r^2 + r^3 + \dots &= \lim_{n \rightarrow \infty} (1 + r + \dots + r^n) \\ &= \lim_{n \rightarrow \infty} \frac{1 - r^{n+1}}{1 - r} \\ &= \frac{1}{1 - r}. \end{aligned}$$

4. The induction begins because $2^4 = 16 < 24 = 4!$. Suppose $k \geq 4$ and the claim holds for $n = k$. Then $2^{k+1} = 2(2^k) < 2(k!) < (k+1)k! = (k+1)!$, establishing the inductive step and completing the proof.

- *5. The induction begins because

$$\frac{1}{1 \times 2} = \frac{1}{2} = \frac{1}{1+1}.$$

Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}, \end{aligned}$$

establishing the inductive step and completing the proof.

6. (a) The induction begins because $1^3 + 5(1) = 6$ is a multiple of 3. Suppose $k \geq 1$ and $k^3 + 5k$ is a multiple of 3, say $k^3 + 5k = 3\ell$. Then

$$(k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5 = 3(\ell + k^2 + k + 2),$$

which is a multiple of 3, establishing the inductive step and completing the proof.

- * (b) The induction begins because $5^1 - 4(1) - 1 = 0$ is a multiple of 16. Suppose $k \geq 1$ and $5^k - 4k - 1$ is a multiple of 16, say $5^k - 4k - 1 = 16\ell$. Then

$$5^{k+1} - 4(k+1) - 1 = 5(5^k - 4k - 1) + 20k + 5 - 4k - 4 - 1 = 5(16\ell) + 16k,$$

which is a multiple of 16, establishing the inductive step and completing the proof.

7. (a) The induction begins because

$$1^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Suppose $k \geq 1$ and the claim holds for $n = k$. Then

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left(k+1 + \frac{k(2k+1)}{6} \right) \\ &= \frac{(k+1)(6k+6+2k^2+k)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}, \end{aligned}$$

establishing the inductive step and completing the proof.

- *(b)** The induction begins because $1^2 = 1^3$. Suppose $k \geq 1$ and the claim holds for $n = k$. Then, using the inductive hypothesis and the formula for the sum of an arithmetic sequence,

$$\begin{aligned} & (1 + 2 + \dots + k + (k + 1))^2 \\ &= (1 + 2 + \dots + k)^2 + 2(k + 1)(1 + 2 + \dots + k) + (k + 1)^2 \\ &= 1^3 + 2^3 + \dots + k^3 + 2(k + 1)\frac{k}{2}(k + 1) + (k + 1)^2 \\ &= 1^3 + 2^3 + \dots + k^3 + (k + 1)^2k + (k + 1)^2 \\ &= 1^3 + 2^3 + \dots + k^3 + (k + 1)^3, \end{aligned}$$

establishing the inductive step and completing the proof.

****8.** *Advanced assignment question: solution withheld.*

****9.** The error is that, in the inductive step argument, it is implicitly assumed that $k \geq 2$, in order that P_2 be common to the two subsets of people. However the inductive hypothesis is only that $k \geq 1$ and this implicit assumption fails when $k = 1$. (If the claim could be checked when $k = 2$ then the argument would become watertight.)

****10.** The induction begins because a $2^1 \times 2^1$ grid with one subsquare missing is simply one L-shape. Suppose $k \geq 1$ and all $2^k \times 2^k$ grids with exactly one subsquare missing can be tiled by L-shapes. Consider any $2^{k+1} \times 2^{k+1}$ grid G with exactly one subsquare missing. Denote the quarters of G by UL for upper-left, UR for upper right, LL for lower left and LR for lower right. Each quarter is a $2^k \times 2^k$ grid, and one of these quarters has one subsquare missing. By rotating G if necessary, without loss of generality, we may suppose that the subsquare missing is in LL . Let T be the L-shape formed by the lower-rightmost subsquare of UL , the lower-leftmost subsquare of UR and the upper-leftmost subsquare of LR . Then removing T from G produces a union of four $2^k \times 2^k$ grids each with one subsquare missing, and each of which can be tiled by L-shapes, by the inductive hypothesis. Hence G is tiled by L-shapes and T , all of which are L-shapes, establishing the inductive step and completing the proof.

11. (a) $a_n = 3(7^n)$. (b) $a_n = 2(5^n)$. (c) $a_n = 4(-1)^n$.

12. (a) The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$ with roots 2 and 3, so the general solution is $a_n = C_1(2^n) + C_2(3^n)$ for some constants C_1, C_2 . But

$$2 = a_0 = C_1 + C_2 \quad \text{and} \quad 5 = a_1 = 2C_1 + 3C_2 .$$

Solving yields $C_1 = C_2 = 1$, so that the final solution is

$$a_n = 2^n + 3^n .$$

- (b) The characteristic equation is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$ with roots 1 and 3, so the general solution is $a_n = C_1(1^n) + C_2(3^n)$ for some constants C_1, C_2 . But

$$-1 = a_0 = C_1 + C_2 \quad \text{and} \quad 2 = a_1 = C_1 + 3C_2 .$$

Solving yields $C_1 = -5/2$ and $C_2 = 3/2$, so that the final solution is

$$a_n = \frac{1}{2}(3^{n+1} - 5) .$$

- *(c) The characteristic equation is $\lambda^2 + 1 = 0$ with roots i and $-i$, where $i = \sqrt{-1}$, so the general solution is $a_n = C_1(i^n) + C_2(-i)^n$ for some constants C_1, C_2 . But

$$4 = a_0 = C_1 + C_2 \quad \text{and} \quad 6 = a_1 = iC_1 - iC_2 .$$

Solving yields $C_1 = 2 - 3i$ and $C_2 = 2 + 3i$, so that the final solution is

$$a_n = (2 - 3i)i^n + (2 + 3i)(-i)^n .$$

- 13.** (a) The characteristic equation is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$ with single root 2, so the general solution is $a_n = C_1(2^n) + C_2n(2^n)$ for some constants C_1, C_2 . But

$$1 = a_0 = C_1 \quad \text{and} \quad 4 = a_1 = 2C_1 + 2C_2 ,$$

yielding $C_1 = C_2 = 1$, so that the final solution is

$$a_n = 2^n(1 + n) .$$

- (b) The characteristic equation is $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$ with single root 3, so the general solution is $a_n = C_1(3^n) + C_2n(3^n)$ for some constants C_1, C_2 . But

$$2 = a_0 = C_1 \quad \text{and} \quad -3 = a_1 = 3C_1 + 3C_2 ,$$

yielding $C_1 = 2$ and $C_2 = -3$, so that the final solution is

$$a_n = 3^n(2 - 3n) .$$

- ** (c) The characteristic equation is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0$ with single root 2, so the general solution is

$$a_n = C_12^n + C_2n2^n + C_3n^22^n$$

for some constants C_1, C_2, C_3 . But

$$2 = a_0 = C_1, \quad 4 = a_1 = 2C_1 + 2C_2 + 2C_3, \quad 16 = a_2 = 4C_1 + 8C_2 + 16C_3 ,$$

yielding $C_1 = 2$, $C_2 = -1$ and $C_3 = 1$, so that the final solution is

$$a_n = 2^n(2 - n + n^2).$$

14. (a) The initial condition is that $a_0 = 100$, and the recurrence is

$$a_{n+1} = \frac{201}{200} a_n + 10 \quad \text{for } n \geq 0.$$

*(b) There are several ways to get a formula. For example, one could plug values into the formula in **1(c)**. A direct calculation is as follows:

$$\begin{aligned} a_n &= \left(\frac{201}{200}\right)^n 100 + 10 \left(\left(\frac{201}{200}\right)^{n-1} + \dots + \left(\frac{201}{200}\right)^2 + \frac{201}{200} + 1 \right) \\ &= \left(\frac{201}{200}\right)^n 100 + 10 \left(\frac{1 - \left(\frac{201}{200}\right)^n}{1 - \frac{201}{200}} \right) \\ &= \left(\frac{201}{200}\right)^n 100 + \left(\frac{201}{200}\right)^n 2,000 - 2,000 \end{aligned}$$

which simplifies to

$$a_n = 2,100 \left(\frac{201}{200}\right)^n - 2,000.$$

Check by induction: $100 = 2,100 \left(\frac{201}{200}\right)^0 - 2,000$ starts the induction.

Using the inductive hypothesis (when $n = k$), we get

$$\begin{aligned} a_{k+1} &= \frac{201}{200} a_k + 10 = \frac{201}{200} \left(2,100 \left(\frac{201}{200}\right)^k - 2,000 \right) + 10 \\ &= 2,100 \left(\frac{201}{200}\right)^{k+1} - 2010 + 10 = 2,100 \left(\frac{201}{200}\right)^{k+1} - 2,000 \end{aligned}$$

which verifies the inductive step and hence also the general formula.

*(c) After 10 and 20 years the account will contain \$1,820.73 and \$4,951.43 respectively. To become a millionaire we require n years where $a_{12n} = 10^6$ so that

$$n = \frac{1}{12} \frac{\ln\left(\frac{1,002,000}{2,100}\right)}{\ln\left(\frac{201}{200}\right)} = 103.05,$$

so that the account will exceed \$1,000,000 in the first month of the 104th year.

15. (a) After one hour has elapsed there will be $2^{12} = 4,096$ organisms. To get one million organisms we need $5n$ minutes where $2^n \geq 10^6$, so that $n \geq 20$,

or at least 100 minutes. To get 100 million organisms we need $5n$ minutes where $2^n \geq 10^8$, so that $n \geq 27$, or at least 2 hours and 15 minutes.

*(b) A general formula is

$$a_n = \begin{cases} 2^n & \text{if } n \leq 11 \\ 2^n - 2,000(2^{n-11} - 1) & \text{if } n \geq 12. \end{cases}$$

The food remains safe provided

$$2^n - 2,000(2^{n-11} - 1) = 2^n \frac{48}{2,048} + 2,000 \leq 1,000,000$$

that is,

$$n \leq \log_2 \left(\frac{998,000 \times 2048}{48} \right) = 25.3,$$

or, for 2 hours and 5 minutes after the fly landed (an extra 25 minutes compared with the situation where no antibiotic had been activated).

16. (a) $a_2 = 2^2$, $a_3 = 2^3$, $a_4 = 2^5$, $a_5 = 2^8$, $a_6 = 2^{13}$.

*(b) The induction begins because $a_0 = 2^1 = 2^{b_0}$ and $a_1 = 2^1 = 2^{b_1}$. Suppose as inductive hypothesis that $k \geq 2$ and $a_k = 2^{b_k}$ and $a_{k-1} = 2^{b_{k-1}}$. Then, using the fact that $b_{k+1} = b_k + b_{k-1}$,

$$a_{k+1} = a_k \cdot a_{k-1} = 2^{b_k} \cdot 2^{b_{k-1}} = 2^{b_k + b_{k-1}} = 2^{b_{k+1}},$$

establishing the inductive step and completing the proof.

****17.** *Advanced assignment question: solution withheld.*

18. We have $a_0 = 1$ and $a_n = 2a_{n-1}$ if $n \geq 1$. Clearly $a_n = 2^n$.

***19.** We have $a_0 = 1$, $a_1 = 2$ and $a_n = a_{n-1} + a_{n-2}$ if $n \geq 2$. To see this notice that in a line of n people with $n \geq 2$, if the last person is F then there are a_{n-1} possibilities for the line of the first $n-1$ people, whilst if the last person is M then the second last person must be F , so that there are a_{n-2} possibilities for the line of the first $n-2$ people. We get the Fibonacci sequence with the first element deleted.

****20.** (a) The induction begins because

$$A_0 = \frac{1}{2} = -\frac{1}{14} \left(\frac{3}{10} \right)^0 + \frac{4}{7}, \quad B_0 = \frac{1}{2} = \frac{1}{14} \left(\frac{3}{10} \right)^0 + \frac{3}{7}.$$

Suppose the claim holds for $n = k \geq 0$. Then

$$\begin{aligned}
A_{k+1} &= \frac{7}{10}A_k + \frac{4}{10}B_k \\
&= \frac{7}{10} \left(-\frac{1}{14} \left(\frac{3}{10} \right)^k + \frac{4}{7} \right) + \frac{4}{10} \left(\frac{1}{14} \left(\frac{3}{10} \right)^k + \frac{3}{7} \right) \\
&= -\frac{1}{14} \left(\frac{3}{10} \right)^k \left(\frac{7}{10} - \frac{4}{10} \right) + \frac{28}{70} + \frac{12}{70} \\
&= -\frac{1}{14} \left(\frac{3}{10} \right)^{k+1} + \frac{4}{7}
\end{aligned}$$

and

$$\begin{aligned}
B_{k+1} &= \frac{3}{10}B_k + \frac{6}{10}B_k \\
&= \frac{3}{10} \left(-\frac{1}{14} \left(\frac{3}{10} \right)^k + \frac{4}{7} \right) + \frac{6}{10} \left(\frac{1}{14} \left(\frac{3}{10} \right)^k + \frac{3}{7} \right) \\
&= \frac{1}{14} \left(\frac{3}{10} \right)^k \left(\frac{6}{10} - \frac{3}{10} \right) + \frac{12}{70} + \frac{18}{70} \\
&= \frac{1}{14} \left(\frac{3}{10} \right)^{k+1} + \frac{3}{7},
\end{aligned}$$

which establishes the inductive step, and the result.

- (b) In the long term (as $n \rightarrow \infty$), company A gets $4/7$ and company B gets $3/7$ of the market share.
- (c) We have

$$A_{n+1} = \frac{7}{10}A_n + \frac{4}{10}B_n, \quad \text{and} \quad B_{n+1} = \frac{3}{10}A_n + \frac{6}{10}B_n,$$

so that

$$B_n = \frac{10}{4} \left(A_{n+1} - \frac{7}{10}A_n \right)$$

and

$$\begin{aligned}
A_{n+2} &= \frac{7}{10}A_{n+1} + \frac{4}{10}B_{n+1} \\
&= \frac{7}{10}A_{n+1} + \frac{4}{10} \left(\frac{3}{10}A_n + \frac{6}{10}B_n \right) \\
&= \frac{7}{10}A_{n+1} + \frac{4}{10} \left(\frac{3}{10}A_n + \frac{6}{10} \left(\frac{10}{4} \left(A_{n+1} - \frac{7}{10}A_n \right) \right) \right) \\
&= \frac{13}{10}A_{n+1} - \frac{3}{10}A_n
\end{aligned}$$

yielding

$$A_{n+1} - \frac{13}{10}A_{n+1} + \frac{3}{10}A_n = 0.$$

Its characteristic equation is

$$\lambda^2 - \frac{13}{10}\lambda + \frac{3}{10} = (\lambda - 1)(\lambda - 3/10) = 0$$

with roots 1 and 3/10. Thus, for some constants C_1 and C_2 ,

$$A_n = C_1 + C_2 \left(\frac{3}{10}\right)^n$$

and

$$\begin{aligned} B_n &= \frac{10}{4} \left(C_1 + C_2 \left(\frac{3}{10}\right)^{n+1} - \frac{7}{10} \left(C_1 + C_2 \left(\frac{3}{10}\right)^n \right) \right) \\ &= \frac{10}{4} \left(\frac{3}{10} C_1 + C_2 \left(\frac{3}{10}\right)^n \left(\frac{3}{10} - \frac{7}{10} \right) \right) \\ &= \frac{3}{4} C_1 - \left(\frac{3}{10}\right)^n C_2. \end{aligned}$$

But

$$\frac{1}{2} = A_0 = C_1 + C_2 \quad \text{and} \quad \frac{1}{2} = B_0 = \frac{3}{4}C_1 - C_2.$$

Solving gives

$$C_1 = \frac{4}{7} \quad \text{and} \quad C_2 = -\frac{1}{14}$$

from which the formulae in (a) quickly follow.

- (d) Suppose more generally that the initial market shares are A_0 and B_0 . Notice that $A_0 + B_0 = 1$. In the final part of the derivation of part (c) we get instead

$$A_0 = C_1 + C_2 \quad \text{and} \quad B_0 = \frac{3}{4}C_1 - C_2$$

which have solutions

$$C_1 = \frac{4}{7} \quad \text{and} \quad C_2 = A_0 - \frac{4}{7}.$$

Thus

$$A_n = \frac{4}{7} + \left(A_0 - \frac{4}{7} \right) \left(\frac{3}{10} \right)^n \rightarrow \frac{4}{7} \quad \text{as} \quad n \rightarrow \infty$$

and

$$B_n = \frac{3}{4} \left(\frac{4}{7} \right) - \left(A_0 - \frac{4}{7} \right) \left(\frac{3}{10} \right)^n \rightarrow \frac{3}{7} \quad \text{as} \quad n \rightarrow \infty.$$

Thus the long term behaviour is the same for all possible initial market shares.