The University of Sydney

MATH2069/2969 Discrete Mathematics and Graph Theory

Semester 1 **Discrete Mathematics Tutorial 2 (Week 3)** 2006

Starred questions are suitable for students enrolled in MATH2969 or for students aiming for a credit or higher.

1. This problem solves a nonhomogeneous recurrence in stages.

(a) Solve the homogeneous recurrence $a_n = 3a_{n-1} + 4a_{n-2}$.

(b) Find a particular solution $p_n = An + B$ of the recurrence

$$a_n \;=\; 3a_{n-1} + 4a_{n-2} - 12n - 2$$

and use part (a) to write down the general solution.

(c) Solve the recurrence in (b) given also that $a_0 = 2$, $a_1 = 3$.

2. Use the method of the previous problem to solve the following recurrences:

(a) $a_n = 5a_{n-1} - 6a_{n-2} + 2n + 3$ for $n \geq 2$ where $a_0 = 2$, $a_1 = 5$.

*(b) $a_n = 5a_{n-1} - 6a_{n-2} + 4^n + 2n + 3$ for $n \geq 2$ where $a_0 = 5$, $a_1 = 19$.

(c) $a_n = 4a_{n-1} - 4a_{n-2} + 2$ for $n \geq 2$ where $a_0 = -1$, $a_1 = 2$.

*(d) $a_n = 4a_{n-1} - 4a_{n-2} + 3^n - 6n + 5$ for $n \geq 2$ where $a_0 = 0$, $a_1 = -2$.

**(e) $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ for $n \geq 2$ where $a_0 = a_1 = 1$.

3. Find simple expressions for the generating functions of the following sequences:

(a) $1, 3, 3^2, 3^3, \ldots$ (b) $1, -1, 1, -1, \ldots$ (c) $2, 2, 2, \ldots$ *(d) $1, -2, 3, -4, \ldots$

4. Write down the sequences associated with the following generating functions:

(a) $(3 + 4z)^2$ (b) $\dfrac{1}{1 + 2z}$ (c) $\dfrac{z}{1 - z}$ **(d) $\dfrac{3z - 1}{(1 + z)^2}$

5. Let $G(z)$ be the generating function of the sequence defined by $a_0 = 5$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$. Verify that $G(z)(1 - 2z) = \dfrac{5 - 4z}{1 - z}$ and deduce the formula $a_n = 6(2^n) - 1$.

****6.** Verify that $\dfrac{z(1 + z)}{(1 - z)^3} = \displaystyle\sum_{n=0}^{\infty} n^2 z^n$.

****7.** Let $G(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $H(z) = \displaystyle\sum_{n=0}^{\infty} (a_0 + a_1 + \cdots + a_n) z^n$. Verify that

$$H(z) \;=\; \frac{G(z)}{1 - z}$$

and use the previous exercise to discover a formula for

$$1^2 + 2^2 + \cdots + n^2 .$$

8. Place the following functions of N in order of increasing growth:

$$N^2\,,\ N^3\,,\ N^{3/2}\,,\ \sqrt{N}\,,\ N^{1/3}\,,\ 2^N\,,\ 10^N\,,\ N^N\,,\ 1.00001^N\,,$$

$$\log N\,,\ N\log N\,,\ N^2\log N\,,\ \log(\log N)\,,\ (\log N)^2\,.$$

9. With just one application of L'Hopital's Rule, verify that N^p grows faster than $\ln N$ for all p such that $0 < p < 1$.

***10.** Fix a positive integer n. Assuming that the limits in the following equation exist, verify by induction that

$$\lim_{x\to\infty} \frac{x}{(\ln x)^n} \;=\; \frac{(n-k)!}{n!} \lim_{x\to\infty} \frac{x}{(\ln x)^{n-k}}$$

for $k = 0, \ldots, n$. Deduce that

$$\lim_{x\to\infty} \frac{x}{(\ln x)^n} \;=\; \infty\,.$$

This tells us that linear growth is faster than all powers of logarithmic growth.

***11.** Let K, L be positive constants and $f(N), g(N)$ be functions of a nonnegative integer N. Make sense of, and then prove, the following statement:

$$O(Kf(N) + Lg(N)) \;=\; O(f(N) + g(N))\,.$$

This enables one to throw away (or introduce) any constants when manipulating a sum using Big-Oh notation.

***12.** Verify that $10^N = O(N!)$ but $N! \neq O(10^N)$.

***13.** Let X be a real number. Define the *floor of* X, denoted by $\lfloor X \rfloor$, to be the largest integer $\leq X$. Define the *ceiling of* X, denoted by $\lceil X \rceil$, to be the smallest integer $\geq X$.

(a) Verify that, except for $N = 1$,

$$\frac{N}{3} \;\leq\; \left\lfloor \frac{N}{2} \right\rfloor$$

for all nonnegative integers N.

(b) Verify that for all integers N and all positive integers M

$$\left\lceil \frac{N}{M} \right\rceil \;=\; \left\lfloor \frac{N+M-1}{M} \right\rfloor .$$

This is a handy device for passing between floors and ceilings.

**14. Verify that the functions $\log N!$ and $N \log N$ grow at the same rate, in the sense of being Big-Oh of each other.

**15. In this problem we justify the claim that if a machine with running time $O(N \log N)$ is sped up λ times (λ being some constant > 1) then for large inputs the maximum size of inputs that can be processed in a given time goes up by approximately the same factor λ. (For machines with $O(N)$ running time the factor is exactly λ.) The problem is expressed in terms of real valued functions and limits.

(a) Let y be defined implicitly as a function of x by the equation

$$y \ln y \;=\; \lambda x \ln x$$

where $\ln x$ denotes the natural logarithm of x. Differentiate and find a formula for the derivative y' in terms of y and x.

(b) Apply L'Hopital's Rule to prove that

$$\lim_{x \to \infty} \frac{y}{x} \;=\; \lambda\,.$$

**16. Consider the sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \ldots$$

Prove that the nth element is

$$\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor .$$