

1. (a) The characteristic equation is  $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$  with roots 4 and  $-1$  so the solution is

$$a_n = C_1(4^n) + C_2(-1)^n$$

for some constants  $C_1, C_2$ .

- (b) Observe that

$$\begin{aligned} An + B &= p_n = 3p_{n-1} + 4p_{n-2} - 12n - 2 \\ &= 3(A(n-1) + B) + 4(A(n-2) + B) - 12n - 2 \\ &= 7An - 11A + 7B - 12n - 2 \end{aligned}$$

so that

$$-6An + 11A - 6B = -12n - 2.$$

Equating coefficients yields  $A = 2$  and  $B = 4$ , so that the particular solution is  $p_n = 2n + 4$ . Adding this to the solution in (a) gives the general solution

$$a_n = C_1(4^n) + C_2(-1)^n + 2n + 4$$

for some constants  $C_1, C_2$ .

- (c) Here  $2 = a_0 = C_1 + C_2 + 4$  and  $3 = a_1 = 4C_1 - C_2 + 6$ . Solving simultaneously yields  $C_1 = -1 = C_2$ , producing the final solution

$$a_n = 2n + 4 - 4^n + (-1)^{n+1}.$$

2. (a) The roots of the characteristic equation are 2 and 3 so the complementary function is  $c_n = C_1(2^n) + C_2(3^n)$ . For a particular solution try  $p_n = An + B$ , so that

$$\begin{aligned} An + B &= 5(A(n-1) + B) - 6(A(n-2) + B) + 2n + 3 \\ &= -An + 7A - B + 2n + 3 \end{aligned}$$

yielding

$$2An - 7A + 2B = 2n + 3.$$

Equating coefficients yields  $A = 1$  and  $B = 5$ , so that  $p_n = n + 5$ . The general solution is

$$a_n = c_n + p_n = C_1(2^n) + C_2(3^n) + n + 5.$$

But  $2 = a_0 = C_1 + C_2 + 5$  and  $5 = a_1 = 2C_1 + 3C_2 + 6$ . Solving simultaneously yields  $C_1 = -8, C_2 = 5$ , producing the final solution

$$a_n = n + 5 - 2^{n+3} + 5(3^n).$$

- \*(b) The complementary function is again  $c_n = C_1(2^n) + C_2(3^n)$ . For a particular solution try  $p_n = An + B + C(4^n)$ , so that

$$\begin{aligned} An + B + C(4^n) &= 5(A(n-1) + B + C(4^{n-1})) \\ &\quad - 6(A(n-2) + B + C(4^{n-2})) + 4^n + 2n + 3 \\ &= -An + 7A - B + 14C(2^{n-2}) + 16(4^{n-2}) + 2n + 3 \end{aligned}$$

yielding

$$2An - 7A + 2B + 2C(4^{n-2}) = 2n + 3 + 16(4^{n-2}).$$

Equating coefficients yields  $A = 1$ ,  $B = 5$ ,  $C = 8$ , so that  $p_n = n + 5 + 2(4^{n+1})$ . The general solution is

$$a_n = c_n + p_n = C_1(2^n) + C_2(3^n) + n + 5 + 2(4^{n+1}).$$

But  $5 = a_0 = C_1 + C_2 + 13$  and  $19 = a_1 = 2C_1 + 3C_2 + 38$ . Solving simultaneously yields  $C_1 = -5$ ,  $C_2 = -3$ , producing the final solution

$$a_n = n + 5 - 5(2^n) - 3^{n+1} + 2(4^{n+1}).$$

- (c) The only root of the characteristic equation is 2 so the complementary function is  $c_n = C_1(2^n) + C_2n(2^n)$ . For a particular solution try  $p_n = A$ , so that  $A = 4A - 4A + 2$ , yielding  $A = 2$ . The general solution is

$$a_n = c_n + p_n = C_1(2^n) + C_2n(2^n) + 2.$$

But  $-1 = a_0 = C_1 + 2$  and  $2 = a_1 = 2C_1 + 2C_2 + 2$ , yielding  $C_1 = -3$ ,  $C_2 = 3$ , producing the final solution

$$a_n = 2^n(3n - 3) + 2.$$

- \*(d) Again the complementary function is  $c_n = C_1(2^n) + C_2n(2^n)$ . For a particular solution try  $p_n = An + B + C(3^n)$ , so that

$$\begin{aligned} An + B + C(3^n) &= 4A(n-1) + 4B + 4C(3^{n-1}) - 4A(n-2) - 4B \\ &\quad - 4C(3^{n-2}) + 3^n - 6n + 5 \\ &= 8C(3^{n-2}) + 4A + 3^n - 6n + 5 \end{aligned}$$

yielding

$$An + B + C(3^{n-2}) = 4A + 9(3^{n-2}) - 6n + 5.$$

Equating coefficients gives  $A = -6$ ,  $B = -19$  and  $C = 9$ . The general solution is

$$a_n = c_n + p_n = C_1(2^n) + C_2n(2^n) + 3^{n+2} - 6n - 19.$$

But  $0 = a_0 = C_1 - 10$  and  $-2 = a_1 = 2C_1 + 2C_2 + 2$ , yielding  $C_1 = 10$ ,  $C_2 = -12$ , producing the final solution

$$a_n = 2^{n+1}(5 - 6n) + 3^{n+2} - 6n - 19.$$

\*\**(e)* Once again the complementary function is  $c_n = C_1(2^n) + C_2n(2^n)$ . For a particular solution try  $p_n = An^2(2^n)$ , so that

$$\begin{aligned} An^2(2^n) &= 4A(n-1)^2(2^{n-1}) - 4A(n-2)^2(2^{n-2}) + 2^n \\ &= 2^{n-2}(4An^2 - 8A + 4) \end{aligned}$$

yielding  $0 = -8A + 4$  so that  $A = 1/2$ . The general solution is

$$a_n = c_n + p_n = C_1(2^n) + C_2n(2^n) + n^22^{n-1}.$$

But  $1 = a_0 = C_1$  and  $1 = a_1 = 2C_1 + 2C_2 + 1$ , yielding  $C_1 = 1$ ,  $C_2 = -1$ , producing the final solution

$$a_n = 2^{n-1}(2 - 2n + n^2).$$

3. (a)  $G(z) = \sum_{n=0}^{\infty} 3^n z^n = \frac{1}{1-3z}$       (b)  $G(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}$

(c)  $G(z) = \sum_{n=0}^{\infty} 2z^n = 2 \sum_{n=0}^{\infty} z^n = \frac{2}{1-z}$

\**(d)* We have  $G(z) = \sum_{n=0}^{\infty} (-1)^n (n+1)z^n$ . Antidifferentiating yields

$$H(z) = C + \sum_{n=0}^{\infty} (-1)^n z^{n+1} = C - \sum_{m=1}^{\infty} (-1)^m z^m.$$

We may take  $C = -1$  so that

$$H(z) = - \sum_{n=0}^{\infty} (-1)^n z^n = \frac{-1}{1+z} = -(1+z)^{-1}$$

giving

$$G(z) = H'(z) = (1+z)^{-2} = \frac{1}{(1+z)^2}.$$

4. (a)  $(3+4z)^2 = 9 + 24z + 16z^2$  so the sequence is  $9, 24, 16, 0, 0, \dots$

(b)  $\frac{1}{1+2z} = 1 - 2z + 2^2z^2 - 2^3z^3 + \dots$ , so the sequence is  $1, -2, 2^2, -2^3, \dots$

(c)  $\frac{z}{1-z} = z + z^2 + z^3 + \dots$ , so the sequence is  $0, 1, 1, 1, \dots$

\*\* (d) Note that from (d) of the previous question,

$$\frac{1}{(1+z)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n,$$

so that

$$\begin{aligned} \frac{3z-1}{(1+z)^2} &= \sum_{n=0}^{\infty} (-1)^n 3(n+1) z^{n+1} - \sum_{n=0}^{\infty} (-1)^n (n+1) z^n \\ &= -1 + \sum_{m=1}^{\infty} (-1)^{m-1} 3m z^m - \sum_{n=1}^{\infty} (-1)^n (n+1) z^n \\ &= -1 + \sum_{n=1}^{\infty} (-1)^{n-1} (3n+n+1) z^n \\ &= \sum_{n=0}^{\infty} (-1)^{n-1} (4n+1) z^n, \end{aligned}$$

so the sequence is  $-1, 5, -9, 13, -17, \dots$

## 5. Using summation notation

$$\begin{aligned} G(z)(1-2z) &= (1-2z) \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} 2a_n z^{n+1} \\ &= \sum_{n=0}^{\infty} a_n z^n - \sum_{m=1}^{\infty} 2a_{m-1} z^m = a_0 + \sum_{m=1}^{\infty} (a_m - 2a_{m-1}) z^m \\ &= a_0 + \sum_{m=1}^{\infty} z^m = 5 - 1 + \sum_{m=0}^{\infty} z^m \\ &= 4 + \frac{1}{1-z} = \frac{5-4z}{1-z}. \end{aligned}$$

Using partial fractions

$$G(z) = \frac{A}{1-z} + \frac{B}{1-2z}$$

where

$$A(1-2z) + B(1-z) = 5-4z.$$

Putting  $z = 1$  yields  $-A = 1$  so  $A = -1$ ; putting  $z = 1/2$  yields  $B/2 = 3$  so  $B = 6$ . Hence

$$G(z) = \frac{-1}{1-z} + \frac{6}{1-2z} = -\sum_{n=0}^{\infty} z^n + 6 \sum_{n=0}^{\infty} 2^n z^n = \sum_{n=0}^{\infty} (6(2^n) - 1) z^n,$$

yielding finally

$$a_n = 6(2^n) - 1.$$

\*\*6. Advanced assignment question: solution withheld.

\*\*7. Observe that

$$\begin{aligned}\frac{G(z)}{1-z} &= (a_0 + a_1z + \dots + a_nz^n + \dots)(1 + z + \dots + z^n + \dots) \\ &= a_0 + (a_0 + a_1)z + \dots + (a_0 + a_1 + \dots + a_n)z^n + \dots \\ &= \sum_{n=0}^{\infty} (a_0 + a_1 + \dots + a_n)z^n = H(z).\end{aligned}$$

Hence, by the previous exercise,

$$\sum_{n=0}^{\infty} (0^2 + 1^2 + \dots + n^2)z^n = \frac{\sum_{n=0}^{\infty} n^2 z^n}{1-z} = \frac{z(1+z)}{(1-z)^4}.$$

But differentiating

$$\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n$$

yields

$$\frac{1}{(1-z)^4} = \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} z^n$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty} (0^2 + 1^2 + \dots + n^2)z^n &= z(1+z) \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} z^n \\ &= \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} z^{n+1} + \sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{6} z^{n+2} \\ &= \sum_{m=1}^{\infty} \frac{(m+2)(m+1)m}{6} z^m + \sum_{m=2}^{\infty} \frac{(m+1)m(m-1)}{6} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+2)(m+1)m}{6} z^m + \sum_{m=0}^{\infty} \frac{(m+1)m(m-1)}{6} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+2)(m+1)m + (m+1)m(m-1)}{6} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+1)m(m+2+m-1)}{6} z^m \\ &= \sum_{m=0}^{\infty} \frac{(m+1)m(2m+1)}{6} z^m = \sum_{n=0}^{\infty} \frac{(n+1)n(2n+1)}{6} z^m\end{aligned}$$

yielding, finally,

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

8. The order of increasing growth is

$$\log(\log N), \log N, (\log N)^2, N^{1/3}, \sqrt{N}, N \log N, N^{3/2}, N^2, \\ N^2 \log N, N^3, 1.00001^N, 2^N, 10^N, N^N.$$

The easiest way to get this is to guess a list and then check that the limit of each successive quotient is  $\infty$ . Then one is sure the order is correct. Most of the limits are trivial or straightforward. Others are facilitated by L'Hopital's Rule.

9. We have

$$\lim_{N \rightarrow \infty} \frac{N^p}{\ln N} = \lim_{N \rightarrow \infty} \frac{pN^{p-1}}{1/N} = p \lim_{N \rightarrow \infty} N^p = \infty.$$

\*10. Observe that

$$\frac{(n-0)!}{n!} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^{n-0}} = \lim_{n \rightarrow \infty} \frac{x}{(\ln x)^n},$$

which starts the induction. Suppose that  $0 \leq k < n$ . Then, by an inductive hypothesis, followed by L'Hopital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^n} &= \frac{(n-k)!}{n!} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^{n-k}} \\ &= \frac{(n-k)!}{n!} \lim_{x \rightarrow \infty} \frac{1}{(n-k)(\ln x)^{n-k-1}(1/x)} \\ &= \frac{(n-k-1)!}{n!} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^{n-k-1}} \\ &= \frac{(n-(k+1))!}{n!} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^{n-(k+1)}} \end{aligned}$$

establishing the inductive step. Hence the result is proved. In particular, taking  $k = n$ ,

$$\lim_{x \rightarrow \infty} \frac{x}{(\ln x)^n} = \frac{1}{n!} \lim_{x \rightarrow \infty} \frac{x}{(\ln x)^0} = \frac{1}{n!} \lim_{x \rightarrow \infty} x = \infty.$$

\*11. The statement means that if a function is eventually bounded by a constant multiple of a linear combination of functions, then it is eventually bounded by a constant multiple of their sum. To prove this, suppose  $C$  and  $N_0$  are positive constants and  $h(N)$  a function of  $N$  such that

$$h(N) \leq C(Kf(n) + Lg(N))$$

for all  $N \geq N_0$ . Then

$$\begin{aligned} h(N) &\leq CKf(n) + CLg(N) \\ &\leq \max\{CK, CL\}f(n) + \max\{CK, CL\}g(n) \\ &= \max\{CK, CL\}(f(n) + g(n)) \end{aligned}$$

for all  $N \geq N_0$ . But  $\max\{CK, CL\}$  is just another positive constant, so this verifies  $h(n) = O(f(n) + g(n))$ .

**\*12.** For  $N \geq 10$ ,

$$\begin{aligned} 10^N &= 10^{N-9} \times 10^9 \\ &\leq 10 \times 11 \times \dots \times (N-1) \times N \times 10^9 \\ &= \frac{10^9}{9!} N! \end{aligned}$$

which verifies that  $10^N = O(N!)$ . Suppose that  $N! = O(10^N)$ , so there are positive constants  $K, N_0$  such that

$$N! \leq K10^N \quad \text{for } N \geq N_0.$$

Choose any  $N$  bigger than  $N_0$  and  $K10^{10}$ . Then

$$\begin{aligned} N! &= N \times (N-1) \times (N-2) \times \dots \times 10 \times 9! \\ &\geq N \times 10 \times 10 \times \dots \times 10 \times 9! = N10^{N-10}9! \\ &> K10^{10}10^{N-10} = K10^N, \end{aligned}$$

which contradicts that  $N! \leq K10^N$ . Hence  $N! \neq O(10^N)$ .

**\*13.** (a) Certainly  $1/3 > 0 = \lfloor 1/2 \rfloor$ , so the inequality fails when  $N = 1$ . If  $N$  is even then  $N/3 \leq N/2 = \lfloor N/2 \rfloor$ . If  $N \geq 3$  is odd then

$$\left\lfloor \frac{N}{2} \right\rfloor = \frac{N-1}{2} = \frac{N}{3} + \frac{N}{6} - \frac{1}{2} \geq \frac{N}{3} + \frac{3}{6} - \frac{1}{2} = \frac{N}{3}$$

and the claim is proved.

(b) Here  $N \in \mathbb{Z}$  and  $M \in \mathbb{Z}^+$ . Observe that  $\frac{N}{M} = X + \frac{Y}{M}$  for some  $Y \in \{0, \dots, M-1\}$ , so that

$$\left\lfloor \frac{N}{M} \right\rfloor = \begin{cases} X & \text{if } Y = 0 \\ X+1 & \text{if } Y > 0. \end{cases}$$

If  $Y = 0$  then

$$\left\lfloor \frac{N+M-1}{M} \right\rfloor = \left\lfloor \frac{N}{M} + \frac{M-1}{M} \right\rfloor = \left\lfloor X + \frac{M-1}{M} \right\rfloor = X = \left\lfloor \frac{N}{M} \right\rfloor$$

since  $\frac{M-1}{M} < 1$ . If  $Y > 0$  then

$$\left\lfloor \frac{N+M-1}{M} \right\rfloor = \left\lfloor \frac{N}{M} + 1 - \frac{1}{M} \right\rfloor = \left\lfloor X+1 + \frac{Y-1}{M} \right\rfloor = X+1 = \left\lfloor \frac{N}{M} \right\rfloor$$

since  $0 \leq \frac{Y-1}{M} < 1$ .

**\*\*14.** *Advanced assignment question: solution withheld.*

**\*\*15.** (a) Implicit differentiation yields

$$(1 + \ln y)y' = \lambda(1 + \ln x)$$

so that

$$y' = \lambda \frac{1 + \ln x}{1 + \ln y}.$$

(b) We have, by two applications of L'Hopital's Rule,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{y}{x} &= \lim_{x \rightarrow \infty} \lambda \frac{1 + \ln x}{1 + \ln y} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{y' \frac{1}{y}} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{y}{xy'} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{y(1 + \ln y)}{\lambda x(1 + \ln x)} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{y + y \ln y}{\lambda x + \lambda x \ln x} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{y + y \ln y}{\lambda x + y \ln y} \\ &= \lambda \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln y} + 1}{\frac{1}{\ln x} + 1} \\ &= \lambda. \end{aligned}$$

**\*\*16.** *Advanced assignment question: solution withheld.*