The University of Sydney

MATH2069/2969 Discrete Mathematics and Graph Theory

Semester 1 **Discrete Mathematics Tutorial 4 (Week 5)** 2006

Starred questions are suitable for students enrolled in MATH2969 or for students aiming for a credit or higher.

1. Denote by P_0 the origin of the xy-plane. The *polar angle* of a point P in the plane is the anticlockwise angle the vector $\overrightarrow{P_0P}$ makes with the positive x-axis.

(a) Put the following fractions in *decreasing* order:

$$\frac{3}{6}\,,\ \frac{1}{1}\,,\ \frac{0.5}{3.5}\,,\ \frac{0}{1}\,,\ \frac{22}{23}\,.$$

Now order the following points by increasing polar angle:

$$P_1(3,6)\,,\quad P_2(1,1)\,,\quad P_3(0.5,3.5)\,,\quad P_4(0,1)\,,\quad P_5(22,23)\,.$$

(b) Use a numerical test to decide whether the points in (a) sorted by increasing polar angle form a convex polygon.

(c) Decide (numerically) whether $Q(15,20)$ lies inside or outside this polygon.

2. Let X be the following list of points in the plane:

$$(6,7)\,,\ (4,10)\,,\ (5,4)\,,\ (2,3)\,,\ (0,5)\,,\ (4,3)\,,\ (1,9)\,,\ (5,8)\,,\ (3,8)\,,\ (4,5)$$

(a) Find the lexicographically least point of X, and subtract its coordinates from every point of X. Write down the new list of points (whose lexicographically least point now is the origin $P_0(0,0)$).

(b) Sort this new list of points by increasing polar angle. If any points have the same polar angle, retain only the point furthest from P_0.

(c) Apply the Graham Scan to your sorted list, to eliminate any points not on the boundary of the convex hull.

(d) Add the coordinates that you subtracted in part (a) to your answers to (c) to produce a final list of points representing the convex hull of X as a polygon in standard form.

3. Find the unique q and r such that $a = qb + r$ where $0 \le r < |b|$ in each of the following cases:

(a) $a = 190\,,\ b = 55.$ (b) $a = 1001\,,\ b = 17.$ (c) $a = -1001\,,\ b = -17.$

*4. Let a, b be integers with $b \neq 0$, so that $a = qb + r$ for unique integers q and r such that $0 \leq r < |b|$. Verify that

$$q = \begin{cases} \lfloor a/b \rfloor & \text{if } b > 0 \\ \lceil a/b \rceil & \text{if } b < 0. \end{cases}$$

5. Use the Euclidean Algorithm to find the greatest common divisor of the following pairs of integers:

(a) $190\,,\ 55\,.$ (b) $1001\,,\ 17\,.$ (c) $100011\,,\ 10011\,.$ (d) $2^{20}\,,\ 20^2\,.$

6. Find the greatest common divisor of 987 and 610. Which numbers arise in the steps of the Euclidean Algorithm? (This is an example of the algorithm going as slow as possible.)

*7. List the equations which appear in the Euclidean Algorithm. Prove by induction that each remainder r_i can be expressed as an integer linear combination of a and b, that is,

$$r_i = ax + by \quad \text{for some integers } x, y\,.$$

In particular deduce that the greatest common divisor of a and b is an integer linear combination of a and b.

*8. Let a, b be integers, not both zero, and let $c = \text{g.c.d.}\,(a, b)$. Suppose d is any divisor of a and b. Certainly $d \leq c$. However, prove that d divides c.

9. Suppose a and b are positive integers. Call a positive integer c the *least common multiple* of a and b, written

$$c = \text{l.c.m.}\,(a, b)$$

if c is a multiple of both a and b, and $c \leq d$ whenever d is a multiple of both a and b.

(a) Find $\text{l.c.m.}\,(2, 3)$, $\text{l.c.m.}\,(4, 12)$, $\text{l.c.m.}\,(4, 10)$, $\text{l.c.m.}\,(6, 15)$.

*(b) Verify that if m is any multiple of both a and b then it is a multiple of $\text{l.c.m.}\,(a, b)$.

*(c) Verify that

$$\text{l.c.m.}\,(a, b)\ \text{g.c.d.}\,(a, b) = ab\,.$$

Give an algorithm, therefore, for finding least common multiples.

*(d) Deduce that if a and b are coprime and c is a multiple of a and b, then c is a multiple of ab.

**10. Let $p \geq 2$ be an integer which is *prime*, that is, has no common divisors other than 1 and p. Let a and b be any integers. Prove that if p divides ab then p divides a or p divides b.