THE UNIVERSITY OF SYDNEY

MATH2069/2969 DISCRETE MATHEMATICS AND GRAPH THEORY

Semester 1 **Discrete Maths Tutorial 4 (Week 5) Solutions** 2006

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1. (a) The following fractions are in decreasing order:

$$\frac{1}{1}\,,\ \frac{22}{23}\,,\ \frac{3}{6}\,,\ \frac{0.5}{3.5}\,,\ \frac{0}{1}\,.$$

Hence the following points are sorted by increasing polar angle:

$$P_2(1,1)\ ,\ \ P_5(22,23)\ ,\ \ P_1(3,6)\ ,\ \ P_3(0.5,3.5)\ ,\ \ P_4(0,1)\ .$$

(b) From (a) we consider the points $P_0, P_2, P_5, P_1, P_3, P_4$ which, to avoid confusion, we rename $Q_0, \ldots, Q_5$. We then test whether $\Delta_{Q_iQ_{i+1}Q_{i+2}} \geq 0$ for $i = 1$ to 3:

$$\begin{aligned}\Delta_{Q_1Q_2Q_3} &= \det\begin{bmatrix}1 & 1 & 1\\ 1 & 22 & 3\\ 1 & 23 & 6\end{bmatrix} = (132-69)-(6-3)+(23-22)\\ &= 63-3+1 = 61 > 0\end{aligned}$$

$$\begin{aligned}\Delta_{Q_2Q_3Q_4} &= \det\begin{bmatrix}1 & 1 & 1\\ 22 & 3 & 0.5\\ 23 & 6 & 3.5\end{bmatrix} = (10.5-3)-(77-11.5)+(132-69)\\ &= 7.5-65.5+63 = 5 > 0\end{aligned}$$

$$\begin{aligned}\Delta_{Q_3Q_4Q_5} &= \det\begin{bmatrix}1 & 1 & 1\\ 3 & 0.5 & 0\\ 6 & 3.5 & 1\end{bmatrix} = (0.5-0)-(3-0)+(10.5-3)\\ &= 0.5-3+7.5 = 5 > 0\end{aligned}$$

The test concludes that $P_0, P_2, P_5, P_1, P_3, P_4$ form the vertices of a convex polygon.

(c) To decide whether $Q(15,20)$ lies inside or outside the polygon, first observe that

$$\frac{22}{23} > \frac{15}{20} > \frac{3}{6}$$

so that Q is located between the rays passing through the origin and the points $P_5(22,23)$ and $P_1(3,6)$. Now we calculate

$$\begin{aligned}\Delta_{P_5QP_1} &= \det\begin{bmatrix}1 & 1 & 1\\ 22 & 15 & 3\\ 23 & 20 & 6\end{bmatrix} = (90-60)-(132-69)+(440-345)\\ &= 30-63+95 = 62 > 0\end{aligned}$$

so that Q lies outside the polygon.

2. (a) The lexicographically least point is (2,3), so our list becomes

$$(4,4)\,,\ (2,7)\,,\ (3,1)\,,\ (0,0)\,,\ (-2,2)\,,\ (2,0)\,,\ (-1,6)\,,\ (3,5)\,,\ (1,5)\,,\ (2,2)$$

(b) First note that the fractions x/y for each point $(x, y) \neq (0,0)$ in our list are sorted in decreasing order as follows:

$$2/0\,,\ 3/1\,,\ 2/2\,,\ 4/4\,,\ 3/5\,,\ 2/7\,,\ 1/5\,,\ -1/6\,,\ -2/2\,.$$

Thus the points are sorted in increasing polar angle (only retaining points furthest from P_0 with the same polar angle) as follows:

$$(2,0)\,,\ (3,1)\,,\ (4,4)\,,\ (3,5)\,,\ (2,7)\,,\ (1,5)\,,\ (-1,6)\,,\ (-2,2)\,.$$

(c) The above points are named P_1 to P_8 in that order and we initialize $I = 2$.
Outer loop $J = 3$:

$$\begin{aligned}\Delta_{P_1P_2P_3} &= \det\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 4\\ 0 & 1 & 4\end{bmatrix} = (12-4)-(8-0)+(2-0)\\ &= 8-8+2 = 2 > 0\end{aligned}$$

so $\triangle P_1P_2P_3$ is anticlockwise; increment $I = 3$.
Outer loop $J = 4$:

$$\begin{aligned}\Delta_{P_2P_3P_4} &= \det\begin{bmatrix}1 & 1 & 1\\ 3 & 4 & 3\\ 1 & 4 & 5\end{bmatrix} = (20-12)-(15-3)+(12-4)\\ &= 8-12+8 = 4 > 0\end{aligned}$$

so $\triangle P_2P_3P_4$ is anticlockwise; increment $I = 4$.
Outer loop $J = 5$:

$$\begin{aligned}\Delta_{P_3P_4P_5} &= \det\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & 2\\ 4 & 5 & 7\end{bmatrix} = (21-10)-(28-8)+(20-12)\\ &= 11-20+8 = -1 < 0\end{aligned}$$

so $\triangle P_3P_4P_5$ is clockwise; decrement $I = 3$, which will delete the point $(3,5)$.

$$\begin{aligned}\Delta_{P_2P_3P_5} &= \det\begin{bmatrix}1 & 1 & 1\\ 3 & 4 & 2\\ 1 & 4 & 7\end{bmatrix} = (28-14)-(21-2)+(12-4)\\ &= 14-19+8 = 3 > 0\end{aligned}$$

so $\triangle P_2P_3P_5$ is anticlockwise; reassign $P_4 = (2,7)$; increment $I = 4$.
Outer loop $J = 6$:

$$\begin{aligned}\Delta_{P_3P_4P_6} &= \det\begin{bmatrix}1 & 1 & 1\\ 4 & 2 & 1\\ 4 & 7 & 5\end{bmatrix} &= (10-7)-(20-4)+(28-8)\\ && = 3-16+20 = 7 > 0\end{aligned}$$

so $\triangle P_3P_4P_6$ is anticlockwise; reassign $P_5 = (1,5)$; increment $I = 5$.
Outer loop $J = 7$:

$$\begin{aligned}\Delta_{P_4P_5P_7} &= \det\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 7 & 5 & 6\end{bmatrix} &= (6+5)-(12+7)+(10-7)\\ && = 11-19+3 = -5 < 0\end{aligned}$$

so $\triangle P_4P_5P_7$ is clockwise; decrement $I = 4$, which will delete the point $(1,5)$.

$$\begin{aligned}\Delta_{P_3P_4P_7} &= \det\begin{bmatrix}1 & 1 & 1\\ 4 & 2 & -1\\ 4 & 7 & 6\end{bmatrix} &= (12+7)-(24+4)+(28-8)\\ && = 19-20+20 = 19 > 0\end{aligned}$$

so $\triangle P_3P_4P_7$ is anticlockwise; reassign $P_5 = (-1,6)$; increment $I = 5$.
Outer loop $J = 8$:

$$\begin{aligned}\Delta_{P_4P_5P_8} &= \det\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -2\\ 7 & 6 & 2\end{bmatrix} &= (-2+12)-(4+14)+(12+7)\\ && = 10-18+19 = 11 > 0\end{aligned}$$

so $\triangle P_4P_5P_8$ is anticlockwise; reassign $P_6 = (-2,2)$; increment $I = 6$.
Halt and output the new $P_0, P_1, \ldots, P_6$:

$$(0,0)\,,\ (2,0)\,,\ (3,1)\,,\ (4,4)\,,\ (2,7)\,,\ (-1,6)\,,\ (-2,2)\ .$$

(d) Adding the coordinates $(2,3)$ yields the convex hull of X as a polygon in standard form:

$$(2,3)\,,\ (4,3)\,,\ (5,4)\,,\ (6,7)\,,\ (4,10)\,,\ (1,9)\,,\ (0,5)\ .$$

3. (a) Observe that $190 = 3(55) + 25$, so $q = 3$ and $r = 25$.

(b) Observe that $1001 = 58(17) + 15$, so $q = 58$ and $r = 15$.

(c) Observe that $-1001 = 59(-17) + 2$, so $q = 59$ and $r = 2$.

***4.** Consider first $b > 0$. We have

$$\frac{a}{b} = \left\lfloor \frac{a}{b} \right\rfloor + c$$

for some c satisfying $0 \leq c < 1$, giving

$$a = \left\lfloor \frac{a}{b} \right\rfloor b + cb$$

where we have $0 \leq cb < b$, giving $r = cb$ and $q = \lfloor a/b \rfloor$.

Now consider $b < 0$. We have

$$\frac{a}{b} = \left\lceil \frac{a}{b} \right\rceil - c$$

for some c satisfying $0 \leq c < 1$, giving

$$a = \left\lceil \frac{a}{b} \right\rceil b - cb$$

where we have $0 \leq -cb < -b$, giving $r = -cb$ and $q = \lceil a/b \rceil$.

These observations combine to give the formula

$$q = \begin{cases} \lfloor a/b \rfloor & \text{if } b > 0 \\ \lceil a/b \rceil & \text{if } b < 0. \end{cases}$$

5. (a) The algorithm produces the following steps:

$$\begin{aligned} 190 &= 3(55) + 25 \\ 55 &= 2(25) + 5 \\ 25 &= 5(5) + 0 \end{aligned}$$

so the greatest common divisor is 5.

(b) The algorithm produces the following steps:

$$\begin{aligned} 1001 &= 38(17) + 15 \\ 17 &= 15 + 2 \\ 15 &= 7(2) + 1 \\ 2 &= 2(1) + 0 \end{aligned}$$

so the greatest common divisor is 1.

(c) The algorithm produces the following steps:

$$\begin{aligned}
100011 &= 9(10011) + 9912 \\
10011 &= 9912 + 99 \\
9912 &= 100(99) + 12 \\
99 &= 8(12) + 3 \\
12 &= 4(3) + 0
\end{aligned}$$

so the greatest common divisor is 3.

(d) The algorithm produces the following steps:

$$\begin{aligned}
2^{20} = 1048576 &= 2621(400) + 176 \\
400 &= 2(176) + 48 \\
176 &= 3(48) + 32 \\
48 &= 32 + 16 \\
32 &= 2(16) + 0
\end{aligned}$$

so the greatest common divisor is 16.

6. The algorithm produces the following steps:

$$\begin{aligned}
987 &= 610 + 377 \\
610 &= 377 + 233 \\
377 &= 233 + 144 \\
233 &= 144 + 89 \\
144 &= 89 + 55 \\
89 &= 55 + 34 \\
55 &= 34 + 21 \\
34 &= 21 + 13 \\
21 &= 13 + 8 \\
13 &= 8 + 5 \\
8 &= 5 + 3 \\
5 &= 3 + 2 \\
3 &= 2 + 1 \\
2 &= 2(1) + 0
\end{aligned}$$

so the greatest common divisor is 1. The remainders are Fibonacci numbers.

***7.** The equations which appear in the Euclidean Algorithm are

$$\begin{array}{rcll} a & = & q_1 r_1 + r_2 & \qquad (b = r_1) \\ r_1 & = & q_2 r_2 + r_3 & \\ r_2 & = & q_3 r_3 + r_4 & \\ \vdots & & & \\ r_{s-2} & = & q_{s-1} r_{s-1} + r_s & \\ r_{s-1} & = & q_s r_s + 0 & \end{array}$$

We claim that each remainder r_i can be expressed as an integer linear combination of a and b. The induction begins because

$$r_1 \;=\; 0(a) + 1(b) \qquad \text{and} \qquad r_2 \;=\; 1(a) - q_1(b) \; .$$

Suppose as inductive hypothesis that $2 < i \leq s$ and

$$r_{i-1} \;=\; xa + yb \qquad \text{and} \qquad r_{i-2} \;=\; za + wb$$

for some x, y, z, w. But

$$r_{i-2} \;=\; q_{i-1} r_{i-1} + r_i \; ,$$

so

$$\begin{array}{rcl} r_i & = & r_{i-2} - q_{i-1} r_{i-1} \\ & = & za + wb - q_{i-1}(xa + yb) \\ & = & (z - q_{i+1} x)a + (w - q_{i-1} y)b \; , \end{array}$$

which is an integer linear combination of a and b, completing the inductive step. The claim follows now by induction. In particular the greatest common divisor r_s is an integer linear combination of a and b.

***8.** From the previous question,

$$c \;=\; xa + yb$$

for some y, b, and

$$a \;=\; zd \qquad \text{and} \qquad b \;=\; wd$$

for some z, w, so that

$$c \;=\; xzd + ywd \;=\; (xz + yw)d \; ,$$

which is a multiple of d.

9. (a) l.c.m. $(2,3) = 6$, l.c.m. $(4,12) = 12$, l.c.m. $(4,10) = 20$, l.c.m. $(6,15) = 30$.

*(b) Suppose $m = ax = by$ for some x, y. Put $\ell = \text{l.c.m.}\,(a,b)$ so $\ell = az = bw$ for some z, w. But $m = q\ell + r$ for some q, r with $0 \leq r < \ell$, so

$$r \;=\; m - q\ell \;=\; ax - qaz \;=\; a(x - qz)$$

and

$$r \;=\; m - q\ell \;=\; by - qbw \;=\; b(y - qw))\ .$$

Thus r is a multiple of a and b. If $r > 0$ then $r \geq \ell$, a contradiction. Hence $r = 0$, so $m = q\ell$ is a multiple of ℓ.

*(c) Put $\ell = \text{l.c.m.}\,(a,b)$ and $g = \text{g.c.d.}\,(a,b)$. Then $gx = a$, $gy = b$, $\ell = az = bw$ for some x, y, z, w. Hence $ab/g = xb = ya$ is a multiple of a and b, so $ab/g \geq \ell$. By (b), ab/ℓ is an integer and we have $(ab/\ell)z = b$, $(ab/\ell)w = a$, so ab/ℓ divides a and b, whence $ab/\ell \leq g$. Hence $ab \geq \ell g \geq ab$, so $\ell g = ab$.

An algorithm, therefore, for finding least common multiples, is to first apply the Euclidean Algorithm to find g and then compute $\ell = ab/g$.

*(d) If a, b are coprime then in (a), $g = 1$ so $\ell = ab$; but by (b), $\ell | c$, so c is a multiple of ab.

****10.** Suppose $p|ab$, say $ab = pq$. Suppose p does not divide a. Then g.c.d. $(p,a) = 1$ since p is prime. By an earlier question, $1 = ax + by$ for some x, y, so

$$b \;=\; b(ax + py) \;=\; abx + bpy \;=\; pqx + pby \;=\; p(qx + by)\ ,$$

which shows $p|b$.