

Starred questions are suitable for students enrolled in MATH2969 or for students aiming for a credit or higher.

1. In each case use the Euclidean Algorithm backwards to express the greatest common divisor of a and b as an integer linear combination of a and b .

(a) $a = 13, b = 8$. (b) $a = 27, b = 21$.
(c) $a = 64, b = 25$. (d) $a = 2^{20}, b = 20^2$.

2. Use the sieve of Eratosthenes to find all primes less than 100.
- *3. Show that N is composite iff N has a divisor M such that $1 < M \leq \sqrt{N}$. (This explains why we halt at \sqrt{N} in constructing the sieve of Eratosthenes.)

4. In each case below express n in the form $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $p_1 < \dots < p_k$ are primes and $\alpha_1, \dots, \alpha_k$ are positive integers:

(a) $n = 1,000,000$. (b) $n = 576,000$. (c) $n = 100^2 - 98^2$.

5. Today is Monday. What day of the week will it be after 100^{100} days have elapsed?
- *6. It has been predicted that exactly 100^{100} hours from 9 a.m. this Monday a meteor will strike the earth. At what time of the day and on what day of the week will this event happen?

7. Find the following elements in \mathbb{Z}_7 and then in \mathbb{Z}_{11} :

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{5}, \frac{9}{10}, \frac{10}{9}$$

Which of these exist in \mathbb{Z}_{10} ?

- *8. Evaluate

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{9} + \frac{1}{10} \pmod{11}$$

Now evaluate

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-2} + \frac{1}{p-1} \pmod{p}$$

where p is any odd prime.

9. Find b^{-1} modulo a in each case, or explain why b^{-1} modulo a does not exist.

(a) $a = 13, b = 8.$

(b) $a = 8, b = 13.$

(c) $a = 64, b = 25.$

(d) $a = 64, b = 50.$

10. Solve the following equations for the least positive integer x :

(a) $3x = 5 \pmod{7}$

(b) $37x = 52 \pmod{7}$

(c) $7x = 6 \pmod{8}$

(d) $64x = 3 \pmod{25}$

*11. Let $x = d_k d_{k-1} \dots d_2 d_1$ be the base 10 representation of an integer x where d_1, \dots, d_k are digits drawn from $0, \dots, 9$. Explain why

$$x = d_1 + d_2 + \dots + d_k \pmod{9}$$

so, also,

$$x = d_1 + d_2 + \dots + d_k \pmod{3}$$

and

$$x = d_1 - d_2 + d_3 - d_4 + \dots + (-1)^{k-1} d_k \pmod{11}.$$

Thus for example to check whether 57,711 is divisible by 9 or 3 we just add up the digits $5 + 7 + 7 + 1 + 1 = 21$. Since 21 is divisible by 3 but not by 9 the same holds for 57,711. To check whether 219,186,429 is divisible by 11 we find the alternating sum $9 - 2 + 4 - 6 + 8 - 1 + 9 - 1 + 2 = 22$. Since 22 is divisible by 11 so is 219,186,429.

**12. Prove that for all positive integers N there exist N consecutive composite numbers. (Hint: let $p_1 < p_2 < \dots < p_M < p$ be the first $M + 1$ primes where $p \geq N + 2$. Inspect all integers from $(p_1 \dots p_M) + 2$ to $(p_1 \dots p_M) + p - 1$.)