The University of Sydney

MATH2069/2969 Discrete Mathematics and Graph Theory

Semester 1	**Discrete Mathematics Tutorial 6 (Week 7)**	2006

Starred questions are suitable for students enrolled in MATH2969 or for students aiming for a credit or higher.

1. Let $A = \{1,2,3,4\}$, $B = \{2,3,5,6\}$, $C = \{3,4,6,7\}$. Find

$$A \cup B\,,\ A \cup C\,,\ B \cap C\,,\ A \cup (B \cap C)\,,\ (A \cup B) \cap (A \cup C)$$

$$C \backslash A\,,\ C \backslash B\,,\ (C \backslash A) \cup (C \backslash B)\,,\ C \backslash (A \cap B)$$

2. Suppose A and B are sets and $|A| = 140$, $|B| = 92$.

(a) Find $|A \cup B|$ given that $|A \cap B| = 36$.

(b) Find $|A \cap B|$ given that $|A \cup B| = 150$.

*(c) Prove that it is impossible to find another set C such that

$$|C| = 58\,,\ \ |A \cap B| = 32\,,\ \ |A \cap B \cap C| \ = \ 10\,,\ \ |A \cup B \cup C| \ = \ 250\,.$$

*__3.__ How many positive integers ≤ 500 are

(a) divisible by at least one of 5, 7, 11?

(b) divisible by 5 and 7 but not by 3?

(b) divisible by 6, but by neither 5 nor 7?

**__4.__ Knowing that any composite number less than 100 must be divisible by 2, 3, 5 or 7 (the only primes $\leq \sqrt{100}$), use the Principle of Inclusion/Exclusion to reproduce your discovery (in a previous tutorial using the sieve of Eratosthenes) that there are 25 primes less than 100.

5. If there are 40 students in the class, explain why at least two have surnames beginning with the same letter. What can be said if there are 80 students in the class?

*__6.__ Show that for any positive integer n there is a positive multiple of n which has only 0's and 1's in its base 10 expansion.

*__7.__ Prove the following Generalised Pigeonhole Principle: If n objects are placed in k boxes then at least one box contains $\lceil n/k \rceil$ objects.

8. How many six digit numbers can you form which do not begin with 0? How many of these have no repetition of digits?

9. There are 10 chairs in a row and 8 students occupy different chairs. Count the number of possible configurations.

10. You have a deck of 52 playing cards.

(a) How many ways can you choose a hand of 5 cards (unordered selection without repetition)?

(b) How many of these hands contain the ace of spades?

*(c) In how many ways can four hands of 5 cards be given to 4 players seated around the table?

*(d) In how many ways can you select four hands of 5 cards from the deck (unordered selection of unordered selections)?

11. Consider the alphabet $\Sigma = \{a, b, c, d, e, f\}$. How many ways can you choose 4 letters from Σ without repetition? with repetition?

12. Given a large supply of jelly beans of 10 different flavours, what is the number of ways you can make up bags of 10 jelly beans?

****13.** Prove that the number of onto functions from a set of size m to a set of size n, where $m \geq n$, is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m .$$

14. State the converse and contrapositive of each of the following implications:

(a) If the sun is shining we are heading off to the surf.

(b) If I have listened attentively I will do well in the exam.

*(c) It is necessary to have a valid password to log on to the computer.

*(d) Only if my attendance is recorded will I be bothered coming to tutorials.

15. Use truth tables to verify that the following are theorems:

(a) $\big[P \;\wedge\; (P \Rightarrow Q)\big] \;\Rightarrow\; Q$ (b) $\big[\sim Q \;\wedge\; (P \Rightarrow Q)\big] \;\Rightarrow\; \sim P$

*(c) $\big[(P \Rightarrow Q) \;\wedge\; (Q \Rightarrow R)\big] \;\Rightarrow\; (P \Rightarrow R)$

*(d) $\big[(P \vee Q) \;\wedge\; (\sim P \;\vee\; R)\big] \;\Rightarrow\; (Q \vee R)$

***16.** Construct one truth table for each of the following compound propositions.

(a) $(P \vee Q) \vee R$ (b) $(P \wedge Q) \vee R$

(c) $(P \Leftrightarrow Q) \wedge R$ (d) $(P \wedge Q) \Leftrightarrow R$

From your table verify that (c) $\Rightarrow$ (b) $\Rightarrow$ (a), but that none of these implications is reversible. Does (d) imply any of (a), (b) or (c)? Is (d) implied by any of (a), (b) or (c)?

17. Express the following statements about integers using quantifiers and usual mathematical symbols:

(a) The product of two negative integers is positive.

(b) The difference of two negative integers need not be positive.

*(c) The set $\mathbb{Z}$ has no largest or smallest element.

**(d) Any subset consisting of negative integers has a largest element.

18. Rewrite the following formal statements so that no use of negation precedes a quantifier, and simplify if possible:

(a) $\sim \big[(\forall x)(\forall y)\ \ P(x,y)\big]$ (b) $\sim \big[(\forall x)(\exists y)\ \ \sim P(x,y)\ \vee \ \sim Q(x,y)\big]$

*(c) $\sim \Big[\sim \big[(\exists x) \sim P(x)\big] \implies (\forall y)\ \big[Q(x,y) \Rightarrow \sim R(x,y)\big]\Big]$

***19.** Interpret the following statements in simple words, and determine whether they are true when the universe of discourse is $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, $\mathbb{Z}_7$ or $\mathbb{Z}_8$.

(a) $(\forall x)(\exists y)\quad x^2 = y$ (b) $(\forall x)(\exists y)\quad x = y^2$

(c) $\sim \big[(\forall x)(\exists y)\quad x = 2y\big]$ (d) $(\forall x)\big[x \neq 0 \ \Rightarrow\ (\exists y) xy = 1\big]$

(e) $(\exists x)(\exists y)\quad x \neq 0\ \wedge\ y \neq 0\ \wedge\ xy = 0$

****20.** (due to Backhouse) Determine the validity of the following argument:

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore Superman does not exist.