The University of Sydney

MATH2069/2969 Discrete Mathematics and Graph Theory

Semester 1 **Discrete Maths Tutorial 6 (Week 7) Solutions** 2006

1. Clearly

$$A \cup B = \{1,2,3,4,5,6\}$$

$$A \cup C = \{1,2,3,4,6,7\}$$

$$B \cap C = \{3,6\}$$

$$A \cup (B \cap C) = \{1,2,3,4,6\} = (A \cup B) \cap (A \cup C)$$

$$C \backslash A = \{6,7\}$$

$$C \backslash B = \{4,7\}$$

$$(C \backslash A) \cup (C \backslash B) = \{4,6,7\} = C \backslash (A \cap B)$$

2. (a) We have

$$|A \cup B| \ = \ |A| + |B| - |A \cap B| \ = \ 140 + 92 - 36 \ = \ 196\ .$$

(b) In this case we have

$$|A \cap B| \ = \ |A| + |B| - |A \cup B| \ = \ 140 + 92 - 150 \ = \ 82\ .$$

(c) Suppose there exists a set C such that

$$|C| = 58\ ,\ \ |A \cap B| = 32\ ,\ \ |A \cap B \cap C| \ = \ 10\ ,\ \ |A \cup B \cup C| \ = \ 250\ .$$

Then

$$\begin{aligned} 250 &= |A \cup B \cup C| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 140 + 92 + 58 - 32 - |A \cap C| - |B \cap C| + 10 \\ &= 268 - |A \cap C| - |B \cap C| \end{aligned}$$

But both $A \cap C$ and $B \cap C$ contain $A \cap B \cap C$, so each have size at least 10. Thus

$$20 \ \leq \ |A \cap C| + |B \cap C| \ \leq 268 - 250 \ = \ 18\ ,$$

which is impossible. Hence no such set C exists.

***3.** (a) Let A, B, C be the sets of positive integers ≤ 500 which are multiples of 5, 7, 11 respectively. We want $|A \cup B \cup C|$. But $|A| = 100$, $|B| = 71$,

$|C| = 45$, $|A \cap B| = 14$, $|A \cap C| = 9$, $|B \cap C| = 6$, $|A \cap B \cap C| = 1$. By inclusion-exclusion

$$|A \cup B \cup C| \;=\; 100 + 71 + 45 - 14 - 9 - 6 + 1 \;=\; 188\,.$$

(b) In addition let D be the set of integers ≤ 500 which are multiples of 3, so we want

$$|(A \cap B)\backslash D| \;=\; |A \cap B| - |A \cap B \cap D| \;=\; 14 - 4 \;=\; 10\,.$$

(c) Now let E be the set of integers ≤ 500 which are multiples of 6, so we want

$$|E\backslash(A\cup B)| \;=\; |E| - |A\cap E| - |B\cap E| + |A\cap B\cap E| \;=\; 83-16-11+2 \;=\; 58\,.$$

****4.** Using the method of **3**(a) we get that the number of integers ≤ 100 which are divisible by at least one of 2, 3, 5 or 7 must be

$$50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0 \;=\; 78\,.$$

But this includes all composite numbers and 2, 3, 5, 7. Hence there are 74 composite numbers between 2 and 100. Thus there are $99 - 74 = 25$ prime numbers less than 100.

5. There are 26 letters in the alphabet. The function that takes a student to the first letter of his or her surname cannot be one-one if there are 40 students, in which case at least two students have surnames beginning with the same letter.

If there are 80 students in the class then at least 4 students must have surnames beginning with the same letter. To see this, suppose otherwise. Then each of the 26 letters is the first letter of at most 3 surnames, so that there are at most $26 \times 3 = 78 < 80$ students, a contradiction.

***6.** Consider the $n + 1$ numbers

$$1\,,\;\; 11\,,\;\; 111\,,\;\; \cdots\,,\;\; 111\ldots1\,,$$

where the last number is a string of $n+1$ digits. If any of these is a multiple of n then we are done. Suppose none of these is a multiple of n. By the pigeonhole principle at least two must be equal modulo n. Subtracting the smaller from the larger gives an integer of the form $1\ldots10\ldots0$, and again we are done, since this difference is a multiple of n.

***7.** Suppose all k boxes contain fewer than $\lceil n/k \rceil$ objects. Then the total number of objects is

$$n \;\leq\; k\big(\lceil n/k \rceil - 1\big) \;\leq\; k\lceil n/k \rceil - k \;<\; k(n/k + 1) - k \;=\; n\,,$$

so that $n < n$, which is impossible. Hence at least one box contains $\lceil n/k \rceil$ objects.

8. The number of six digit numbers which do not begin with 0 is $9 \times 10^5 = 900,000$. The number which have no repetition is $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080$.

9. We are counting ordered selections without repetition. There are $10_{(8)} = 10!/2! = 1,814,400$ possible configurations.

10. You have a deck of 52 playing cards.

(a) The number of ways of choosing a hand of 5 cards (unordered selection without repetition) is

$$\binom{52}{5} = \frac{52!}{47!5!} = 2,598,960 \; .$$

(b) The number of these hands which contain the ace of spades is

$$\binom{51}{4} = \frac{51!}{47!4!} = 249,900 \; .$$

*(c) The number of ways four hands of 5 cards be given to 4 players seated around the table is

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \frac{52!}{47!5!}\frac{47!}{42!5!}\frac{42!}{37!5!}\frac{37!}{32!5!} = \frac{52!}{32!5!5!5!5!} \; .$$

*(d) There are 4! permutations of the four positions around the table, so the number of ways of selecting four hands of 5 cards from the deck, as an unordered selection of unordered selections, is the previous answer divided by 4!, that is

$$\frac{52!}{32!5!5!5!5!4!} \; .$$

11. Consider the alphabet $\Sigma = \{a, b, c, d, e, f\}$. The number of ways of choosing 4 letters from Σ without repetition is

$$\binom{6}{4} = \frac{6!}{4!2!} = 15 \; .$$

The number of ways of choosing 4 letters from Σ with repetition is

$$\binom{6+4-1}{4} = \binom{9}{4} = \frac{9!}{4!5!} = 126 \; .$$

12. Given a large supply of jelly beans of 10 different flavours, the number of ways you can make up bags of 10 jelly beans is

$$\binom{10+10-1}{10} = \binom{19}{10} = 92,378\ .$$

**13. Let X be the set of all functions from A to B where A has size m and B has size n. For each $b \in B$, put

$$X_b \;=\; \{\, f \in X \mid b \notin \text{image of } f\,\}\ .$$

The number N of onto functions from A to B is therefore

$$N \;=\; \Big| X \backslash \big(\cup_{b\in B}\, X_b \big) \Big|\ .$$

But $|X| \;=\; n^m$, and, for $k = 1, \ldots, m$,

$$|X_{b_1} \cap X_{b_2} \cap \ldots \cap X_{b_k}| \;=\; (n-k)^m$$

for $b_1, \ldots, b_k \in B$ pairwise different. Also there are $\binom{n}{k}$ subsets of B of size k. By inclusion-exclusion,

$$\begin{aligned}
\Big| \big(\cup_{b\in B}\, X_b \big) \Big| \;&=\; \sum_b |X_b| - \sum_{b_1\neq b_2} |X_{b_1} \cap X_{b_2}| \\
&\qquad + \sum_{b_1\neq b_2\neq b_3\neq b_1} |X_{b_1} \cap X_{b_2} \cap X_{b_3}| \;-\; \cdots \\
&=\; \sum_{k=1}^{n} (-1)^{n-1} \binom{n}{k} (n-k)^m\ .
\end{aligned}$$

Thus

$$\begin{aligned}
N \;&=\; |X| - \sum_{k=1}^{n} (-1)^{n-1} \binom{n}{k} (n-k)^m \\
&=\; n^m + \sum_{k=1}^{n} (-1)^{n} \binom{n}{k} (n-k)^m \\
&=\; \sum_{k=0}^{n} (-1)^{n} \binom{n}{k} (n-k)^m\ .
\end{aligned}$$

14. (a) Converse: If we are heading off to the surf then the sun is shining.

Contrapositive: If we are not heading off to the surf then the sun is not shining.

(b) Converse: If I do well in the exam then I have listened attentively.

Contrapositive: If I don't do well in the exam then I have not listened attentively.

*(c) Converse: If I have a valid password then I can log on to the computer.

Contrapositive: If I don't have a valid password then I can't log on to the computer.

*(d) Converse: If my attendance is recorded then I will be bothered coming to tutorials.

Contrapositive: If my attendance is not recorded then I won't be bothered coming to tutorials.

15. (a)

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(b)

P	Q	$P \Rightarrow Q$	$\sim Q \wedge (P \Rightarrow Q)$	$[\sim Q \wedge (P \Rightarrow Q)] \Rightarrow \sim P$
T	T	T	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

*(c)

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Observe that T in the 3rd column is always accompanied by T in the 4th, which shows

$$\big[(P \Rightarrow Q) \;\wedge\; (Q \Rightarrow R)\big] \;\Rightarrow\; (P \Rightarrow R)$$

is a theorem.

*(d)

P	Q	R	$P \vee Q$	$\sim P \vee R$	$(P \vee Q) \wedge (\sim P \vee R)$	$Q \vee R$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	F	F

Observe that T in the 3rd column is always accompanied by T in the 4th, which shows

$$\big[(P \vee Q) \ \wedge \ (\sim P \ \vee \ R)\big] \ \Rightarrow \ (Q \vee R)$$

is a theorem.

***16.** The combined truth table is

P	Q	R	(a)	(b)	(c)	(d)
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	F	F	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	T	F
F	F	F	F	F	F	T

from which it follows that (c) implies (b), because the two T's in the column for (c) are accompanied by T's in the column for (b). Similarly (b) implies (a). However (d) does not imply (a) because (d) is true and (a) is false when each of P, Q, R is false. This tells us immediately also that (d) does not imply (c) or (b).

Also (a), (b) and (c) are all true, whilst (d) is false, when P is false, Q is false and R is false. Hence none of (a), (b) or (c) imply (d).

17. (a) $(\forall x, y \in \mathbb{Z}) \quad (x < 0) \wedge (y < 0) \ \Rightarrow \ xy > 0$.

(b) $\sim \big[(\forall x, y \in \mathbb{Z}) \quad (x < 0) \wedge (y < 0) \ \Rightarrow \ x - y > 0\big]$.

*(c) $\sim \big[(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})\ x \leq y\,\big] \wedge \ \sim \big[(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})\ x \geq y\,\big]$

**(d) $(\forall X \subseteq \mathbb{Z})\ \big[(\forall x \in X)\ x < 0\,\big] \ \Rightarrow \ \big[(\exists y \in X)(\forall z \in X)\ y \geq z\,\big]$

18. (a) $(\exists x)(\exists y) \ \sim P(x, y)$

(b) $(\exists x)(\forall y)\ P(x,y)\ \wedge\ Q(x,y)$

*(c) $(\forall x)P(x)\ \wedge\ (\exists y)\big[Q(x,y)\ \wedge\ R(x,y)\big]$

***19. (a)** The square of every number is a number. This is true in all cases.

(b) Every number has a square root. This is true in $\mathbb{R}^+$ and $\mathbb{C}$ only.

(c) Not every number is twice another number. This is true in $\mathbb{Z}$ and $\mathbb{Z}_8$ only.

(d) Every nonzero element has a multiplicative inverse. This is true in $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$ and $\mathbb{Z}_7$ only.

(e) There exists two nonzero elements whose product is zero. This is true in $\mathbb{Z}_8$ only.

****20.** Let A be the proposition that Superman is able to prevent evil, B that he is willing to prevent evil, C that he is impotent, D that he is malevolent, E that he prevents evil, and F that he exists. The premises are

$$(A \wedge B) \Rightarrow E\ ,\ \ \sim A \Rightarrow C\ ,\ \ \sim B \Rightarrow D\ ,\ \ \sim E\ ,\ \ F \Rightarrow (\sim C \wedge \ \sim D)\ .$$

By Modus Tollens and one of De Morgan's Laws,

$$\sim (A \wedge B)\ \equiv \sim A \vee \sim B\ ,$$

so by two applications of Modus Ponens and De Morgan's Law again,

$$C \vee D\ \equiv \sim (\sim C \wedge \sim D)\ .$$

Feeding in one more application of Modus Tollens gives finally

$$\sim F$$

so, indeed, Superman does not exist!!