THE UNIVERSITY OF SYDNEY

MATH2069/2969 DISCRETE MATHEMATICS

Semester 1 **Quiz 1A** 2006

Name (with surname underlined):

Student No.:

Circle one of: **MATH2069** **MATH2969**

This quiz lasts 35 minutes and is worth 30 marks.

1. Complete the following definitions:

(a) The *Fibonacci numbers* $a_0, a_1, a_2, \ldots$ are defined recursively by $a_0 = a_1 = 1$ and, for $n \geq 2$,

(b) We write $f(N) = O(g(N))$ if there exist positive constants C and N_0 such that, for $N \geq N_0$,

(c) The *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the

$(1 + 1 + 1 = 3$ marks)

2. Place the following functions of a positive integer N in order of increasing growth:

$$N^2\,,\ N^{3/2}\,,\ \sqrt{N}\,,\ \log N\,,\ \log(\log N)\,,\ N \log N\,.$$

Answer:

(2 marks)

3. (a) Find A and B such that $\frac{3-2z}{1-3z+2z^2} = \frac{A}{1-z} + \frac{B}{1-2z}$.

Answers: [ ]

(b) Suppose $a_0, a_1, a_2, \ldots$ has generating function $G(z) = \frac{3-2z}{1-3z+2z^2}$. Find

$a_0 =$ [ ] , $a_1 =$ [ ] , $a_2 =$ [ ] .

(2 + 1 + 1 + 1 = 5 marks)

4. Consider the recurrence

$$a_n = 2a_{n-1} - a_{n-2} + 2^{n+1}.$$

(a) If c_n is the complementary function and p_n is a particular solution, then the general solution is

$a_n =$ [ ]

(b) The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$. Hence the complementary function is

$c_n =$ [ ]

(c) A particular solution of the form $p_n = K(2^n)$ is

$p_n =$ [ ]

(1 + 1 + 2 = 4 marks)

5. Evaluate

(a) $1 + 3 + 5 + 7 + \ldots + 99 =$ [ ]

(b) $1 + 2 + 2^2 + 2^3 + \ldots + 2^{99} =$ [ ]

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + (-1)^n \frac{1}{2^n} + \ldots =$ [ ]

(1 + 2 + 2 = 5 marks)

6. Consider the following proof that, for a positive integer n,

$$2^n \; < \; n!$$

precisely when $n \geq 4$.

Proof:

Note first that **(1)**

$$2^1 = 2 > 1 = 1! \,, \;\; 2^2 = 4 > 2 = 2! \;\text{ and }\; 2^3 = 8 > 6 = 3! \,. \qquad \textbf{(2)}$$

Now observe that $2^4 \; = \; 16 \; < \; 24 \; = \; 4!$. **(3)**

Suppose that $k \geq 4$ and $2^k < k!$. **(4)**

Then

$$2^{k+1} \;=\; 2(2^k) \;<\; 2(k!) \qquad \textbf{(5)}$$

$$< \; (k+1)(k!) \;=\; (k+1)! \qquad \textbf{(6)}$$

Thus the claim for $n \geq 4$ now follows by Mathematical Induction. **(7)**

(a) In which line is the inductive hypothesis stated?

Answer:

(b) Which line starts the induction?

Answer:

(c) In which line is the inductive hypothesis used?

Answer:

(d) In which line of the inductive step is the definition of factorial notation used?

Answer:

(e) Which lines comprise the inductive step?

Answer:

(1 + 1 + 1 + 1 + 1 = 5 marks)

7. Consider each statement. Circle **T** if you believe it is true. Circle **F** if you believe it is false. (If you are unsure, leave it.)

(i) The sequence $3, -3, 3, -3, \ldots$ has generating function

$$G(z) = \frac{3}{1+z} \; .$$ **T** **F**

(ii) The sequence $0, 1, 2, 3, \ldots$ has generating function

$$G(z) = \sum_{m=1}^{\infty} m z^m \; .$$ **T** **F**

(iii) The sequence $1, -3, 3^2, -3^3, 3^4, \ldots$ has generating function

$$G(z) = \frac{1}{1-3z} \; .$$ **T** **F**

(iv) $$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n \; .$$ **T** **F**

(v) $$\frac{1}{(1-2z)^2} = \sum_{n=0}^{\infty} (n+1) 2^n z^n \; .$$ **T** **F**

(vi) If $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is given by

(a) $G(z) = \dfrac{1}{1+z}$ then $a_9 = 1$; **T** **F**

$a_{90} = -1$. **T** **F**

(b) $G(z) = \dfrac{z}{1-3z}$ then $a_{900} = 3^{899}$. **T** **F**

(c) $G(z) = \dfrac{5-7z}{1-3z+2z^2}$ then $a_1 = -7$; **T** **F**

$a_3 = 26$. **T** **F**

(vii) If $\lim_{N\to\infty} \dfrac{f(N)}{g(N)} = \infty$ then $f(N) = O(g(N))$. **T** **F**

(viii) If $\lim_{N\to\infty} \dfrac{f(N)}{g(N)} = \infty$ and $\lim_{N\to\infty} \dfrac{g(N)}{h(N)} = \infty$ then

$$h(N) = O(f(N)) \; .$$ **T** **F**

[1/2 mark for each correct answer,
−1/2 mark for each incorrect answer,
0 mark for each answer unattempted,
maximum total = 6 marks,
minimum total = −6 marks]

END OF QUIZ