THE UNIVERSITY OF SYDNEY

MATH2069/2969 DISCRETE MATHEMATICS

| Semester 1 | **Quiz 1B** | 2006 |
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**Name (with surname underlined):**

**Student No.:**

**Circle one of:** **MATH2069** **MATH2969**

*This quiz lasts 35 minutes and is worth 30 marks.*

**1.** (a) Find $A$ and $B$ such that $\dfrac{1+z}{1-3z+2z^2} = \dfrac{A}{1-z} + \dfrac{B}{1-2z}$ .

**Answers:**

(b) Suppose $a_0, a_1, a_2, \ldots$ has generating function $G(z) = \dfrac{1+z}{1-3z+2z^2}$. Find

$a_0 = \square$ , $a_1 = \square$ , $a_2 = \square$ .

$(2 + 1 + 1 + 1 = 5$ marks)

**2.** Evaluate

(a) $1 + 3 + 5 + 7 + \ldots + 99 =$

(b) $1 + 2 + 2^2 + 2^3 + \ldots + 2^{99} =$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + (-1)^n \frac{1}{2^n} + \ldots =$

$(1 + 2 + 2 = 5$ marks)

**3.** Consider the following proof that, for a positive integer $n$,

$$2^n \quad < \quad n!$$

precisely when $n \geq 4$.

*Proof:*

Note first that $2^1 = 2 > 1 = 1!$ , $2^2 = 4 > 2 = 2!$ and $2^3 = 8 > 6 = 3!$ . **(1)**

Now observe that $2^4 \ = \ 16 \ < \ 24 \ = \ 4!$ . **(2)**

Suppose that $k \geq 4$ and $2^k < k!$ . **(3)**

Then

$$\begin{aligned} 2^{k+1} &= 2(2^k) && \textbf{(4)} \\ &< 2(k!) && \textbf{(5)} \\ &< (k+1)(k!) && \textbf{(6)} \\ &= (k+1)! && \textbf{(7)} \end{aligned}$$

The claim for $n \geq 4$ now follows by Mathematical Induction. **(8)**

(a) Which line starts the induction?

**Answer:**

(b) In which line does the inductive hypothesis appear?

**Answer:**

(c) Between which two lines is the inductive hypothesis used?

**Answer:**

(d) Which lines comprise the inductive step?

**Answer:**

(e) Between which two lines of the inductive step is the definition of factorial notation being used?

**Answer:**

$(1 + 1 + 1 + 1 + 1 = 5$ marks)

**4.** Complete the following definitions:

(a) The *floor* of a real number $x$, denoted by $\lfloor x \rfloor$, is the

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(b) The *Fibonacci numbers* $a_0, a_1, a_2, \ldots$ are defined recursively by $a_0 = a_1 = 1$ and, for $n \geq 2$,

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(b) We write $f(N) = O(g(N))$ if there exist positive constants $C$ and $N_0$ such that, for $N \geq N_0$,

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$(1 + 1 + 1 = 3$ marks$)$

**5.** Place the following functions of a positive integer $N$ in order of increasing growth:

$$N^3\,,\ N^{5/2}\,,\ \sqrt{N}\,,\ (\log N)^2\,,\ \log N\,,\ N^2 \log N\,.$$

**Answer:**

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(2 marks)

**6.** Consider the recurrence $a_n = 2a_{n-1} - a_{n-2} + 4(3^n)$ .

(a) If $c_n$ is the complementary function and $p_n$ is a particular solution, then the general solution is

$a_n =$

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(b) The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ . Hence the complementary function is

$c_n =$

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(c) A particular solution of the form $p_n = K(3^n)$ is

$p_n =$

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$(1 + 1 + 2 = 4$ marks$)$

**7.** Consider each statement. Circle **T** if you believe it is true. Circle **F** if you believe it is false. (If you are unsure, leave it.)

(i) The sequence $2, 2, 2, 2, \ldots$ has generating function $G(z) = \dfrac{2}{1+z}$ . **T** **F**

(ii) The sequence $1, 2, 3, 4, \ldots$ has generating function

$$G(z) = \frac{1}{(1-z)^2} \; .$$ **T** **F**

(iii) The sequence $2, 0, 2, 0, 2, 0, \ldots$ has generating function

$$G(z) \;=\; \sum_{n=0}^{\infty} (1 + (-1)^n) z^n \; .$$ **T** **F**

(iv) $$\frac{1}{1-z} \;=\; \sum_{m=1}^{\infty} z^{m-1} \; .$$ **T** **F**

(v) $$\frac{1}{(1-z)^4} \;=\; \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) z^n \; .$$ **T** **F**

(vi) If $G(z) \;=\; \sum_{n=0}^{\infty} a_n z^n$ is given by

(a) $G(z) \;=\; \dfrac{1}{1-z}$ then $a_7 \;=\; -1$ ; **T** **F**

$a_{101} \;=\; 1$ . **T** **F**

(b) $G(z) \;=\; \dfrac{z}{1-2z}$ then $a_{10} \;=\; 2^{11}$ . **T** **F**

(c) $G(z) \;=\; \dfrac{5-7z}{1-3z+2z^2}$ then $a_0 \;=\; 5$ ; **T** **F**

$a_2 \;=\; 12$ . **T** **F**

(vii) If $\displaystyle\lim_{N\to\infty} \frac{f(N)}{g(N)} = 0$ then $g(N) = O(f(N))$ . **T** **F**

(viii) If $\displaystyle\lim_{N\to\infty} \frac{f(N)}{g(N)} = 0$ and $\displaystyle\lim_{N\to\infty} \frac{g(N)}{h(N)} = 0$ then $f(N) = O(h(N))$ . **T** **F**

[ 1/2 mark for each correct answer,
−1/2 mark for each incorrect answer,
0 mark for each answer unattempted,
maximum total = 6 marks,
minimum total = −6 marks]

**END OF QUIZ**