Ring homomorphisms and the spectrum

Let \( \varphi : A \to B \) be a ring homomorphism. Then \( \varphi \) induces a mapping \( \varphi^* : Y \to X : q \mapsto \varphi^{-1}(q) \), where \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \).

**Theorem**

i) if \( f \in A \), then \( \varphi^{-1}(X_f) = Y_{\varphi(f)} \).

ii) if \( a \) is an ideal of \( A \), then \( \varphi^{-1}(\mathcal{V}(a)) = \mathcal{V}(\varphi(a)) \).

**Corollary**

*The map \( \varphi^* \) is continuous.*
The spectrum and localisation

**Theorem**

Let \( S \) be a multiplicatively closed subset of \( A \) and let \( \varphi : A \to S^{-1}A \) be the natural homomorphism. Then \( \varphi^* : \text{Spec}(S^{-1}A) \to \text{Spec}(A) \) is a homeomorphism of \( \text{Spec}(S^{-1}A) \) onto its image in \( X = \text{Spec}(A) \). Furthermore,

i) if \( f \in A \), the image of \( \text{Spec}(Af) \) in \( X \) is \( X_f \);  
ii) if \( p \in X \), then image of \( \text{Spec}(Ap) \) in \( X \) is the intersection of all the open neighbourhoods of \( p \) in \( X \).

Introduction to modules

Elementary linear algebra is concerned with vector spaces in which the scalars form a **field**.

Instead of a field of scalars we would like to use a ring (such as \( \mathbb{Z} \) or \( F[x] \)) and see to what extent the theory of vector spaces can be imitated.

This leads us to the concept of a module.

It turns out that, in addition to examples reminiscent of vector spaces, ideals and quotient rings are both examples of modules.
The definition of a module

Definition

A module over a ring $A$ (in brief, an $A$-module) is an abelian group $M$ (whose group operation is $+$) together with an action of $A$ on $M$; that is, a map $A \times M \rightarrow M : (a, m) \mapsto am$ such that

1. $a(m_1 + m_2) = am_1 + am_2$, for all $a \in A$ and all $m_1, m_2 \in M$. (M1)
2. $(a_1 + a_2)m = a_1m + a_2m$, for all $a_1, a_2 \in A$ and all $m \in M$. (M2)
3. $(a_1a_2)m = a_1(a_2m)$, for all $a_1, a_2 \in A$ and all $m \in M$. (M3)
4. $1m = m$, for all $m \in M$. (M4)

Three simple consequences of the axioms are:

i) $0m = 0$
ii) $a0 = 0$
iii) $(-a)m = -(am) = a(-m)$

Abelian groups

To say that $M$ is an abelian group means that the operation of addition satisfies:

1. $m_1 + m_2 = m_2 + m_1$, for all $m_1, m_2 \in M$. (AB1)
2. $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$, for all $m_1, m_2, m_3 \in M$. (AB2)
3. There exists $0 \in M$ such that $0 + m = m + 0 = m$, for all $m \in M$. (AB3)
4. For all $m \in M$ there exists $-m \in M$ such that $m + (-m) = 0$. (AB4)

Three simple consequences of the axioms are:

i) $0m = 0$
ii) $a0 = 0$
iii) $(-a)m = -(am) = a(-m)$
Example: a ring is a module

The first four axioms in the definition of a ring are the axioms for an abelian group.

Therefore, a ring together with the operation of addition is also an abelian group. Furthermore, if we take $M = A$ and define the action of $A$ on itself by multiplication: $(a, b) \mapsto ab$, then the module axioms M1, M2, M3 and M4 are the last four axioms in the definition of a ring.

Thus a ring is a module over itself where the “action” is given by multiplication.

Example: vector spaces

If the ring $A$ is a field, the axioms for a module define a *vector space* over $A$. That is, a vector space is none other than a module over a field.
Example: free modules

Given a ring $A$, let $A^n$ denote the set of all $n$-tuples $(r_1, r_2, \ldots, r_n)$ with entries from $A$. Then $A^n$ is a module with addition defined componentwise and the action of $a \in A$ defined by

$$a(r_1, r_2, \ldots, r_n) = (ar_1, ar_2, \ldots, ar_n).$$

The module $A^n$ is called the free module of rank $n$. It behaves very much like a vector space. However, not all modules are this nice. When $n = 1$ we recover the example of a ring as a module over itself.

Example: a single endomorphism of a vector space

Given a field $F$, a vector space $V$ over $F$, and a linear transformation $T : V \to V$ we can make $V$ into an $F[x]$-module by defining the action of the polynomial $f(x) \in F[x]$ on $V$ by

$$f(x)v = f(T)v$$

where $f(T)$ is the linear transformation obtained by substituting $T$ into $f(x)$. For example, if $f(x) = x^2 + 3x - 1$, then $f(T) = T^2 + 3T - I$, where $I$ is the identity transformation.
Example: abelian groups

Every abelian group is a \( \mathbb{Z} \)-module. If \( M \) is an abelian group, \( m \in M \) and \( a \in \mathbb{Z} \), then the action of \( a \) on \( m \) is defined to be

\[
am = \begin{cases} 
m + m + \cdots + m & \text{if } a > 0 \\
a \text{ times} \end{cases}
\]

\[
am = \begin{cases} 
0 & \text{if } a = 0 \\
-(m + m + \cdots + m) & \text{if } a < 0 \\
-n \text{ times} \end{cases}
\]

Example: ideals and quotients of ideals.

If \( A \) is a ring and if \( \alpha \) is an ideal of \( A \), then both \( \alpha \) and \( A/\alpha \) are \( A \)-modules.

In the case of \( A/\alpha \), the action of \( \alpha \in A \) on the coset \( s + \alpha \) is defined to be

\[
a(s + \alpha) = as + \alpha.
\]

Thus a module is a generalisation of many concepts including ideals and of the quotient of a ring by an ideal.
Submodules

Definition

A submodule of an $A$-module $M$ is a non-empty subset $N$ such that

- **SM1:** $n_1 + n_2 \in N$ for all $n_1, n_2 \in N$
- **SM2:** $an \in N$ for all $a \in A$ and $n \in N$.

It follows from SM2 that for $n \in N$ we have $-n = (-1)n \in N$.

Therefore, a submodule $N$ of the module $M$ is a subgroup of the additive group of $M$.

If we consider $A$ as a module over itself as in the Example above, then the submodules of $A$ are the ideals of $A$.

Homomorphisms

Given $A$-modules $M$ and $N$, a homomorphism from $M$ to $N$ is a function $\theta : M \to N$ such that

- **H1:** $\theta(m_1 + m_2) = \theta(m_1) + \theta(m_2)$ for all $m_1, m_2 \in M$.
- **H2:** $\theta(rm) = r\theta(m)$ for all $r \in A$ and all $m \in M$.

The kernel of $\theta$ is the submodule of $M$ given by

$$\ker(\theta) = \{ m \in M \mid \theta(m) = 0 \}.$$  

The image of $\theta$ is the submodule of $N$ given by

$$\text{im}(\theta) = \{ \theta(m) \mid m \in M \}.$$  

As usual, a homomorphism is an isomorphism if it is both one-to-one and onto.