Exact sequences

A sequence of $A$-modules and $A$-homomorphisms

$$
\cdots \rightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \cdots
$$

is said to be exact at $M_i$ if $\text{im}(f_i) = \ker(f_{i+1})$. The sequence itself is exact if it is exact at each $M_i$.

A sequence $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact if and only if $f$ is injective, $g$ is surjective and $\text{im}(f) = \ker(g)$. An exact sequence of this type is called a short exact sequence.
**Exactness of Hom, Part 1**

**Theorem**

The sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{} 0$$

is exact if and only if for all $A$-modules $N$, the sequence

$$0 \xrightarrow{} \text{Hom}(M'', N) \xrightarrow{\bar{g}} \text{Hom}(M, N) \xrightarrow{\bar{f}} \text{Hom}(M', N)$$

is exact.

**Exactness of Hom, Part 2**

**Theorem**

The sequence

$$0 \xrightarrow{} N' \xrightarrow{f} N \xrightarrow{g} N''$$

is exact if and only if for all $A$-modules $M$, the sequence

$$0 \xrightarrow{} \text{Hom}(M, N') \xrightarrow{\bar{f}} \text{Hom}(M, N) \xrightarrow{\bar{g}} \text{Hom}(M, N'')$$

is exact.
Split short exact sequences

**Theorem**

If $0 \longrightarrow M' \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} M'' \longrightarrow 0$ is a short exact sequence, the following are equivalent:

(i) there is an isomorphism $\varphi : M \cong M' \oplus M''$ such that for $m' \in M'$ and $m'' \in M''$ we have $\varphi f (m') = (m', 0)$ and $g \varphi^{-1}(m', m'') = m''$.

(ii) there exists an $A$-homomorphism $s : M'' \rightarrow M$ such that $gs = 1$.

(iii) there exists an $A$-homomorphism $t : M \rightarrow M'$ such that $tf = 1$.

An exact sequence satisfying any (and hence all) of these conditions is called a **split** short exact sequence.

Projective modules

An $A$-module $P$ is **projective** if for all short exact sequences

$$0 \longrightarrow N' \overset{f}{\longrightarrow} N \overset{g}{\longrightarrow} N'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow \text{Hom}(P, N') \overset{\bar{f}}{\longrightarrow} \text{Hom}(P, N) \overset{\bar{g}}{\longrightarrow} \text{Hom}(P, N'') \longrightarrow 0$$

is exact. We say that $\text{Hom}(P, -)$ is exact.

**Theorem**

*Every free $A$-module is projective.*
An $A$-module $I$ is *injective* if for all short exact sequences

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

the sequence

$$0 \to \text{Hom}(M'', I) \xrightarrow{\bar{g}} \text{Hom}(M, I) \xrightarrow{\bar{f}} \text{Hom}(M', I) \to 0$$

is exact. We say that $\text{Hom}(\_ , I)$ is exact.