1. Suppose that $A$ is a PID and that $S$ is a multiplicatively closed subset of $A$. Prove that $S^{-1}A$ is a PID.

2. Suppose that $S$ is a multiplicatively closed subset of a ring $A$ and that $A$ is a subring of $B$. Let $p$ be a prime ideal of $A$ such that $p \cap S = \emptyset$ and let $q$ be a prime ideal of $B$ such that $q \cap S = \emptyset$. Prove that $p = A \cap q$ if and only if $S^{-1}p = S^{-1}A \cap S^{-1}q$.

3. Suppose that $S$ is a multiplicatively closed subset of a ring $A$. Prove that an ideal $a$ of $A$ is contracted (with respect to the natural homomorphism $A \to S^{-1}A$) if and only if $(a : s) = a$ for all $s \in S$. Prove that $a \mapsto S^{-1}a$ is a one-to-one correspondence between the contracted ideals of $A$ and the ideals of $S^{-1}A$.

4. Suppose that $A$ is a ring such that $x^2 = x$ for all $x \in A$ and let $X = \text{Spec}(A)$.
   (i) Prove that $X_f$ is both open and closed in the Zariski topology of $X$.
   (ii) For $f_1, \ldots, f_n \in A$, show that $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.
   (iii) Prove that the sets $X_f$ are the only subsets of $X$ which are open and closed. [Use the facts that every open subset of $X$ is a union of basic open sets and every closed subset is compact.]
   (iv) Prove that $X$ is a compact Hausdorff space.

5. If $M$ is an $A$-module, prove that $\text{Hom}_A(A, M) \to M : \varphi \mapsto \varphi(1)$ is an isomorphism.

6. Let $A$ be an integral domain with field of fractions $K$ and suppose that $f \in A$ is nonzero and not a unit. Let $A[1/f]$ denote the subring of $K$ generated by $A$ and $1/f$. Prove that $A[1/f]$ is not finitely generated by showing that if $A[1/f]$ were to have a finite set of generators, then some finite collection of powers of $1/f$ would generate $A[1/f]$ and then $f$ would be a unit.

7. If $A$ is a local ring, prove that $A^m$ and $A^n$ are isomorphic as $A$-modules if and only if $m = n$. 