1. Suppose that $M$ is an $A$-module, $N$ is a submodule of $N$ and $a$ is an ideal of $A$. Prove that $a(M/N) = (aM + N/N)$.

2. Let $a$ be an ideal of $A$. Suppose that $aM = M$ implies $M = 0$ for all finitely generated $A$-modules $M$. Prove that $a$ is contained in the Jacobson radical of $A$.

3. If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of $A$-modules, prove that if $M'$ and $M''$ are finitely generated, then so is $M$.

4. Suppose that $M_1$ and $M_2$ are submodules of the $A$-module $M$.
   
   (i) Show that the the sequence
   \[ 0 \rightarrow M_1 \cap M_2 \xrightarrow{\phi} M_1 \oplus M_2 \xrightarrow{\psi} M_1 + M_2 \rightarrow 0 \]
   is exact, where $\phi(m) = (m, m)$ for $m \in M_1 \cap M_2$ and $\psi(m_1, m_2) = m_1 - m_2$ for $m_1 \in M_1$ and $m_2 \in M_2$.

   (ii) Show that the the sequence
   \[ 0 \rightarrow M/(M_1 \cap M_2) \xrightarrow{\eta} M/M_1 \oplus M/M_2 \xrightarrow{\theta} M/(M_1 + M_2) \rightarrow 0 \]
   is exact, where $\eta(x + M_1 \cap M_2) = (x + M_1, x + M_2)$ for $x \in M$ and $\theta(y + M_1, z + M_2) = y - z + (M_1 + M_2)$ for $y, z \in M$.

   (iii) If $a$ and $b$ are ideals of $A$, use (ii) to show that $a$ and $b$ are coprime if and only if the map $A \rightarrow A/a \times A/b : x \mapsto (x + a, x + b)$ is surjective.

5. Suppose that $M_1$ and $M_2$ are submodules of an $A$-module $M$. If $M_1 + M_2$ and $M_1 \cap M_2$ are finitely generated, prove that $M_1$ and $M_2$ are finitely generated.

6. Suppose that $A$ is a local ring and that $M$ and $N$ are $A$-modules.

   (i) If $m$ is the maximal ideal of $A$, show that there is a homomorphism from $M \otimes_A N$ onto $(M/mM) \otimes_{A/m} (N/mN)$.

   (ii) If $M$ and $N$ are finitely generated, show that $M \otimes_A N = 0$ if and only if $M = 0$ or $N = 0$. 