MATH1902 Linear Algebra

Lecture 1
Week 1, Semester 1, 2001

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Lecture Notes: *Vectors* by C. J. Durrant
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Objectives

- be able to identify and distinguish between **scalar** and **vector** quantities.

- know what is meant by the **position vector** of a point and illustrate this with a diagram.

- be able to explain how to add vectors using either the **triangle rule** or the **parallelogram rule**.

- be able to use position vectors to solve problems in **geometry**.

- know what it means for a point to **divide** a line segment in a given **ratio**.

Scalars and Vectors

A **scalar** is a real number. (Towards the end of the course we will allow complex numbers as well.)

Examples: mass, volume, temperature, density, electric charge, work, etc.

A **vector** is a quantity with a **length** and a **direction**.

Examples: velocity, acceleration, momentum, force, electric and magnetic field intensities.

A vector is represented by a **directed line segment**.

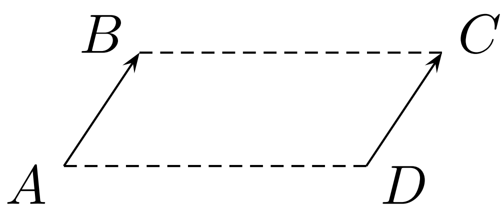
The **length** of a vector $\mathbf{v}$ is written $|\mathbf{v}|$.

Position Vectors

Given points A and B, the **position vector** of B relative to A is the vector, written $\overrightarrow{AB}$ represented by the line segment starting at A and ending at B.

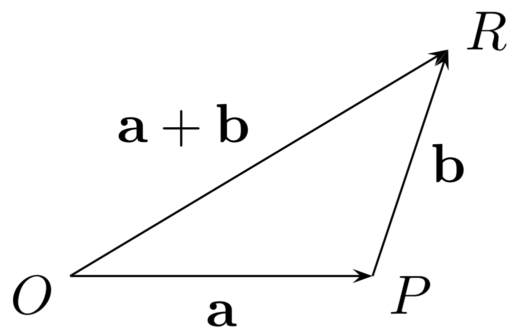
If we write $\mathbf{v} = \overrightarrow{AB}$, then we can think of $\mathbf{v}$ as the vector which **acts** on A to get B.

If the line segments AB and DC have the same direction and the same length, then $ABCD$ is a parallelogram and the position vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal; we write this as $\overrightarrow{AB} = \overrightarrow{DC}$.

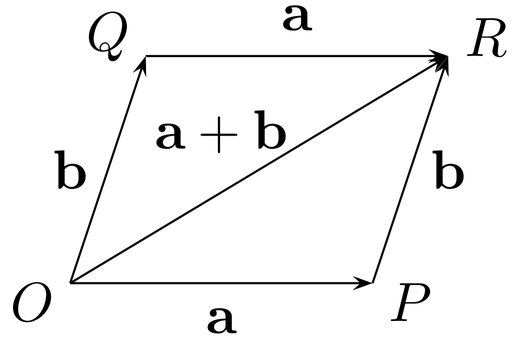


Vector addition

Given vectors $\mathbf{a}$ and $\mathbf{b}$ we can **add** them as follows. First choose a point O, then let P be the point such that $\overrightarrow{OP} = \mathbf{a}$ and finally let R be the point such that $\overrightarrow{PR} = \mathbf{b}$. Then $\mathbf{a} + \mathbf{b} = \overrightarrow{OR}$. This is the **triangle rule** for addition of vectors.



Alternatively, we can represent $\mathbf{b}$ by the line segment from O to Q which is parallel to PR and the same length as PR. This picture is known as the **parallelogram rule** for addition.



Notice that the triangle rule gives:
$\overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$.

We can turn this around to obtain the **head minus tail** rule: $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$.

Negatives

Each vector $\mathbf{v}$ has a **negative**. This is the vector with the same length but the opposite direction. If $\mathbf{v} = \overrightarrow{AB}$, then $-\mathbf{v} = \overrightarrow{BA}$.

The vector which represents the position of A with respect to itself is the **zero vector $\mathbf{0}$**. That is, $\mathbf{0} = \overrightarrow{AA}$.

Multiplication by a scalar

Given a positive scalar s and a vector $\mathbf{v}$, the vector $s\mathbf{v}$ has the same direction as $\mathbf{v}$ but its length is $s|\mathbf{v}|$.

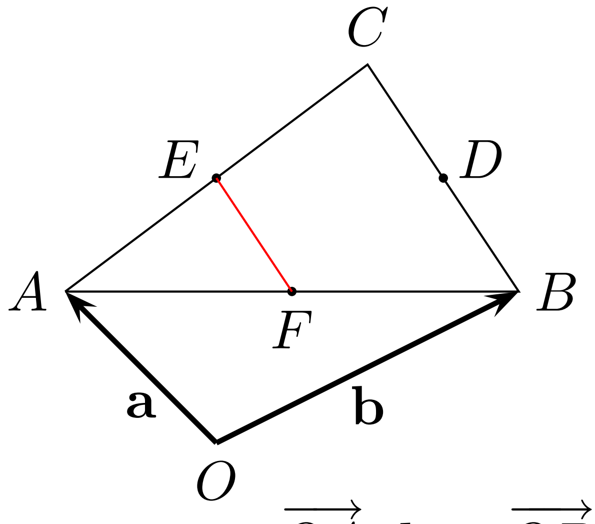
If s is negative, then $s\mathbf{v}$ has the **opposite** direction to $\mathbf{v}$. In general, we have $|s\mathbf{v}| = |s|\,|\mathbf{v}|$, where $|s|$ is the **absolute value** of s and defined by

$$|s| = \begin{cases} s & \text{if } s \geq 0 \\ -s & \text{if } s < 0 \end{cases}$$

Geometry and Vectors

Let ABC be a triangle and let D, E and F be the mid-points of the sides BC, CA and AB. Show that $\overrightarrow{FE} = \frac{1}{2}\overrightarrow{BC}$ and that the sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$ is the zero vector.

Choose a point in the plane and call it O. All our position vectors will be with respect to O.



In particular, we put $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$.

Then $\overrightarrow{OE} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$. But we know that $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ and therefore $\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.

Similarly, $\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and therefore

$$\begin{aligned}\overrightarrow{FE} &= \overrightarrow{OE} - \overrightarrow{OF} \\ &= \frac{1}{2}(\mathbf{a}+\mathbf{c}) - \frac{1}{2}(\mathbf{a}+\mathbf{b}) \\ &= \frac{1}{2}(\mathbf{c}-\mathbf{b}) = \frac{1}{2}\overrightarrow{BC}.\end{aligned}$$

For the last part, we have

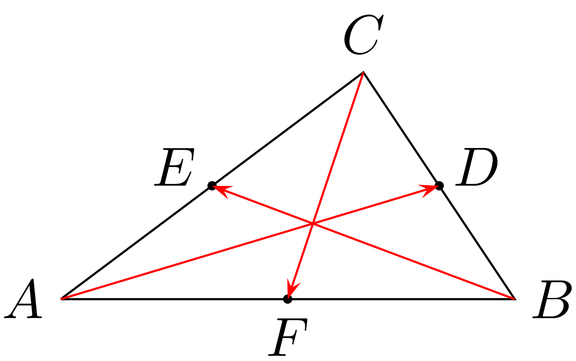
$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \frac{1}{2}(\mathbf{b}+\mathbf{c}) - \mathbf{a}$,

$\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = \frac{1}{2}(\mathbf{a}+\mathbf{c}) - \mathbf{b}$, and

$\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \frac{1}{2}(\mathbf{a}+\mathbf{b}) - \mathbf{c}$.

From this we calculate that

$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \mathbf{0}$.



Division of a line segment in a given ratio (internal)

Suppose that we want to find the point P that divides a given line segment AB internally in the ratio $m : n$. This means that

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{PB}|} = \frac{m}{n}.$$

Because $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction we can write $\overrightarrow{AP} = \frac{m}{n}\overrightarrow{PB}$ and, by the 'head minus tail' rule, $\overrightarrow{PB} = \overrightarrow{AB} - \overrightarrow{AP}$. Therefore $\overrightarrow{AP} = \frac{m}{n}(\overrightarrow{AB} - \overrightarrow{AP})$ and solving for $\overrightarrow{AP}$ gives $\overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$.

Given a point O, we can find the formula for the position vector $\overrightarrow{OP}$. We apply the 'head minus tail' rule to the equation $n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$ to obtain $n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$ and then solve for $\overrightarrow{OP}$. That is,

$$\overrightarrow{OP} = \frac{n\,\overrightarrow{OA} + m\,\overrightarrow{OB}}{m+n}.$$

Division of a line segment in a given ratio (general case)

The formula

$$\overrightarrow{OP} = \frac{n\,\overrightarrow{OA} + m\,\overrightarrow{OB}}{m+n}$$

makes sense even when one of m or n is negative.

If one of m or n is negative, then P is on the line joining A to B, but outside the segment between A and B.

In all cases, if P divides AB in the ratio $m : n$, then we have $n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$.

