MATH1902 Linear Algebra

Lecture 1
Week 1, Semester 1, 2001

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Lecture Notes: Vectors by C. J. Durrant
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Objectives

• be able to identify and distinguish between scalar and vector quantities.

• know what is meant by the position vector of a point and illustrate this with a diagram.

• be able to explain how to add vectors using either the triangle rule or the parallelogram rule.

• be able to use position vectors to solve problems in geometry.

• know what it means for a point to divide a line segment in a given ratio.
Scalars and Vectors

A **scalar** is a real number. (Towards the end of the course we will allow complex numbers as well.)

Examples: mass, volume, temperature, density, electric charge, work, etc.

A **vector** is a quantity with a **length** and a **direction**.

Examples: velocity, acceleration, momentum, force, electric and magnetic field intensities.

A vector is represented by a **directed line segment**.

The **length** of a vector \( \mathbf{v} \) is written \( |\mathbf{v}| \).
Position Vectors

Given points $A$ and $B$, the position vector of $B$ relative to $A$ is the vector, written $\overrightarrow{AB}$ represented by the line segment starting at $A$ and ending at $B$.

If we write $\mathbf{v} = \overrightarrow{AB}$, then we can think of $\mathbf{v}$ as the vector which acts on $A$ to get $B$.

If the line segments $AB$ and $DC$ have the same direction and the same length, then $ABCD$ is a parallelogram and the position vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal; we write this as $\overrightarrow{AB} = \overrightarrow{DC}$.
Vector addition

Given vectors $\mathbf{a}$ and $\mathbf{b}$ we can add them as follows. First choose a point $O$, then let $P$ be the point such that $\mathbf{OP} = \mathbf{a}$ and finally let $R$ be the point such that $\mathbf{PR} = \mathbf{b}$. Then $\mathbf{a} + \mathbf{b} = \mathbf{OR}$. This is the triangle rule for addition of vectors.

Alternatively, we can represent $\mathbf{b}$ by the line segment from $O$ to $Q$ which is parallel to $PR$ and the same length as $PR$. This picture is known as the parallelogram rule for addition.

Notice that the triangle rule gives:

$$\mathbf{OP} + \mathbf{PR} = \mathbf{OR}.$$

We can turn this around to obtain the head minus tail rule: $\mathbf{PR} = \mathbf{OR} - \mathbf{OP}$. 

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Negatives

Each vector \( \mathbf{v} \) has a **negative**. This is the vector with the same length but the opposite direction. If \( \mathbf{v} = \overrightarrow{AB} \), then \( -\mathbf{v} = \overrightarrow{BA} \).

The vector which represents the position of \( A \) with respect to itself is the **zero vector** \( \mathbf{0} \). That is, \( \mathbf{0} = \overrightarrow{AA} \).

**Multiplication by a scalar**

Given a positive scalar \( s \) and a vector \( \mathbf{v} \), the vector \( s\mathbf{v} \) has the same direction as \( \mathbf{v} \) but its length is \( s|\mathbf{v}| \).

If \( s \) is negative, then \( s\mathbf{v} \) has the **opposite** direction to \( \mathbf{v} \). In general, we have \( |s\mathbf{v}| = |s||\mathbf{v}| \), where \( |s| \) is the **absolute value** of \( s \) and defined by

\[
|s| = \begin{cases} 
  s & \text{if } s \geq 0 \\
  -s & \text{if } s < 0
\end{cases}
\]
Geometry and Vectors

Let $ABC$ be a triangle and let $D$, $E$ and $F$ be the mid-points of the sides $BC$, $CA$ and $AB$. Show that $\overrightarrow{FE} = \frac{1}{2} \overrightarrow{BC}$ and that the sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$ is the zero vector.

Choose a point in the plane and call it $O$. All our position vectors will be with respect to $O$.

In particular, we put $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$.

Then $\overrightarrow{OE} = \overrightarrow{OA} + \frac{1}{2} \overrightarrow{AC}$. But we know that $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ and therefore $\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.

Similarly, $\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and
therefore

\[
\overrightarrow{FE} = \overrightarrow{OE} - \overrightarrow{OF} = \frac{1}{2}(a + c) - \frac{1}{2}(a + b) = \frac{1}{2}(c - b) = \frac{1}{2}\overrightarrow{BC}.
\]

For the last part, we have

\[
\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \frac{1}{2}(b + c) - a,
\]

\[
\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = \frac{1}{2}(a + c) - b, \text{ and}
\]

\[
\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \frac{1}{2}(a + b) - c.
\]

From this we calculate that

\[
\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \mathbf{0}.
\]
Division of a line segment in a given ratio (internal)

Suppose that we want to find the point \( P \) that divides a given line segment \( AB \) internally in the ratio \( m : n \). This means that

\[
\frac{|AP|}{|PB|} = \frac{m}{n}.
\]

Because \( \overrightarrow{AP} \) and \( \overrightarrow{PB} \) have the same direction we can write \( \overrightarrow{AP} = \frac{m}{n} \overrightarrow{PB} \) and, by the ‘head minus tail’ rule, \( \overrightarrow{PB} = \overrightarrow{AB} - \overrightarrow{AP} \). Therefore

\[
\overrightarrow{AP} = \frac{m}{n} \left( \overrightarrow{AB} - \overrightarrow{AP} \right)
\]

and solving for \( \overrightarrow{AP} \) gives

\[
\overrightarrow{AP} = \frac{m}{m+n} \overrightarrow{AB}.
\]

Given a point \( O \), we can find the formula for the position vector \( \overrightarrow{OP} \). We apply the ‘head minus tail’ rule to the equation \( n \overrightarrow{AP} = m \overrightarrow{PB} \) to obtain

\[
n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})
\]

and then solve for \( \overrightarrow{OP} \). That is,

\[
\overrightarrow{OP} = \frac{n \overrightarrow{OA} + m \overrightarrow{OB}}{m + n}.
\]
Division of a line segment in a given ratio (general case)

The formula

\[ \overrightarrow{OP} = \frac{n \overrightarrow{OA} + m \overrightarrow{OB}}{m + n} \]

makes sense even when one of \( m \) or \( n \) is negative.

If one of \( m \) or \( n \) is negative, then \( P \) is on the line joining \( A \) to \( B \), but outside the segment between \( A \) and \( B \).

In all cases, if \( P \) divides \( AB \) in the ratio \( m : n \), then we have \( n \overrightarrow{AP} = m \overrightarrow{PB} \).