MATH1902 Linear Algebra

Lecture 2
Week 1, Semester 1, 2001

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Lecture Notes: *Vectors* by C. J. Durrant
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Objectives

- be able to use the laws of vector algebra to simplify expressions involving vectors and scalars
- know what it means for a point to **divide** a line segment in a given **ratio** $m : n$ when one of m or n is negative.
- be able to use **Cartesian** coordinates
- be able to use **polar** coordinates

Vector algebra

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$
$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$$
$$s(\mathbf{a}+\mathbf{b})=s\mathbf{a}+s\mathbf{b}$$
$$(s+t)\mathbf{a}=s\mathbf{a}+t\mathbf{a}$$
$$s(t\mathbf{a})=(st)\mathbf{a}$$
$$1\,\mathbf{a}=\mathbf{a}$$

There is a vector $\mathbf{0}$ (called the **zero vector**) with the property that for all vectors $\mathbf{a}$

$$\mathbf{a}+\mathbf{0}=\mathbf{a}.$$

To each vector $\mathbf{a}$ there corresponds a vector $-\mathbf{a}$ such that $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$.

From these axioms we can deduce that $0\mathbf{a}=\mathbf{0}$ and $(-1)\,\mathbf{a}=-\mathbf{a}$.

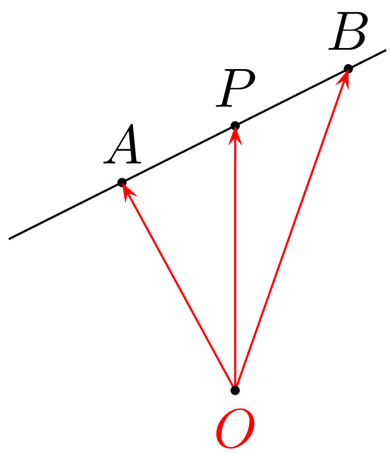
The difference $\mathbf{a}-\mathbf{b}$ of two vectors is defined to be the sum $\mathbf{a}+(-\mathbf{b})$.

More about division of a line segment

Recall from last lecture that if m and n are non-negative and if P divides the line segment AB in the ratio $m : n$, then P lies on the segment between A and B and for any point O we have

$$\overrightarrow{OP} = \frac{n\,\overrightarrow{OA} + m\,\overrightarrow{OB}}{m+n}$$

If $m = 0$, then $P = A$ and if $n = 0$, then $P = B$. This immediately shows us that we had better exclude the possibility that $m = n = 0$.

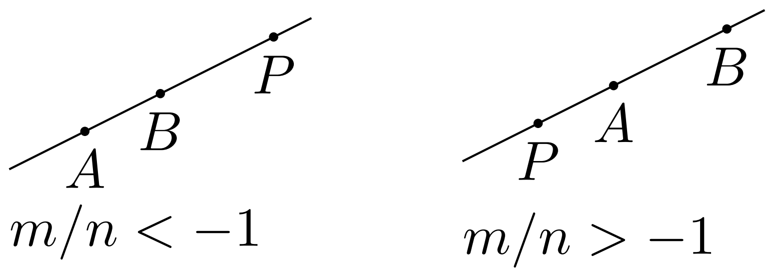


The case $m/n < 0$

In this case we may suppose that n is positive and that m is negative. By definition we have $\overrightarrow{AP} = \frac{m}{n}\,\overrightarrow{PB}$, and so $\overrightarrow{AP} = (-\frac{m}{n})\,\overrightarrow{BP}$. This shows that P is on the line through A and B and there are two cases:

1. if $-\frac{m}{n} > 1$, then B is between A and P.

2. if $-\frac{m}{n} < 1$, then A is between B and P.



The calculations of the previous lecture lead to the same equation:

$$\overrightarrow{OP} = \frac{n\,\overrightarrow{OA} + m\,\overrightarrow{OB}}{m+n}$$

What happens if $\dfrac{m}{n} = -1$?

Example

Let ABC be a triangle. Suppose that P divides AB in the ratio $2:3$ and that Q divides AC in the ratio $3:4$. Let X be the point of intersection of BQ and CP. Show that X divides BQ in the ratio $21:8$.

Solution. (Draw a picture.)

We have $\overrightarrow{OP} = \frac{1}{5}(3\overrightarrow{OA} + 2\overrightarrow{OB})$ and $\overrightarrow{OQ} = \frac{1}{7}(4\overrightarrow{OA} + 3\overrightarrow{OC})$.

For some s we have

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OP} + s\overrightarrow{PC} \\ &= \overrightarrow{OP} + s(\overrightarrow{OC} - \overrightarrow{OP}) \\ &= (1-s)\overrightarrow{OP} + s\overrightarrow{OC}\end{aligned}$$

and similarly, for some t

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OQ} + t\overrightarrow{QB} \\ &= \overrightarrow{OQ} + t(\overrightarrow{OB} - \overrightarrow{OQ}) \\ &= (1-t)\overrightarrow{OQ} + t\overrightarrow{OB}\end{aligned}$$

Equating these two expressions for $\overrightarrow{OX}$ and using the expressions for $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ obtained above, we get

$$\frac{(1-s)}{5}(3\overrightarrow{OA}+2\overrightarrow{OB})+s\overrightarrow{OC}= \frac{(1-t)}{7}(4\overrightarrow{OA}+3\overrightarrow{OC})+t\overrightarrow{OB}$$

We can use the fact that O can be any point to our advantage: we choose it to be A and then all the vectors $\overrightarrow{OA}$ are $\mathbf{0}$. That is,

$$\frac{2(1-s)}{5}\overrightarrow{AB}+s\overrightarrow{AC}=\frac{3(1-t)}{7}\overrightarrow{AC}+t\overrightarrow{AB}$$

But $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have different directions. Thus

$$\frac{2(1-s)}{5}=t \quad \text{and}$$
$$s=\frac{3(1-t)}{7}$$

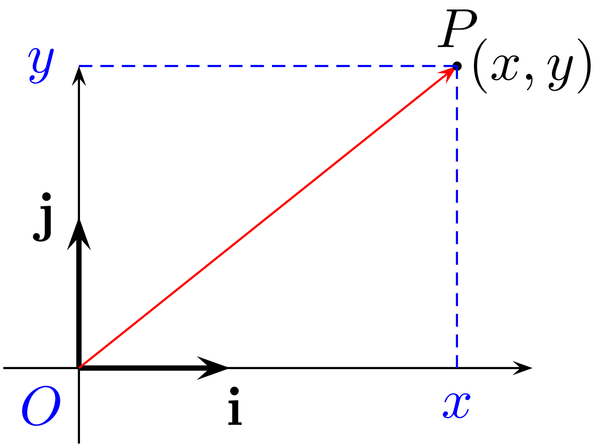
If we substitute the value of s given in the second equation into the first equation, we find that $t = 8/29$. Thus

$$\overrightarrow{OX} = \frac{8\,\overrightarrow{OB} + 21\,\overrightarrow{OQ}}{29}$$

That is, X divides BQ in the ratio $21 : 8$.

Cartesian coordinates

Choose a point O in the plane to be the origin, let $\mathbf{i}$ be a vector of length 1 and let $\mathbf{j}$ be a vector of length 1 that is perpendicular to $\mathbf{i}$.



Given a point P in the plane, with coordinates (x, y) relative to the above coordinate system, let $\mathbf{r}$ be the position vector of P with respect to O. Then, if A is the point with coordinates $(x, 0)$ and if B is the point with coordinates $(0, y)$, then by the parallelogram rule we have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = x\,\mathbf{i} + y\,\mathbf{j}.$$

By Pythagoras' Theorem we have $|\mathbf{r}|^2 = x^2 + y^2$. We often write r for the magnitude of $\mathbf{r}$. That is,

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2}.$$

Polar coordinates

Instead of using the pair of numbers (x, y) to represent the point $\mathbf{r} = \overrightarrow{OP}$ we use $r = |\overrightarrow{OP}|$ and the angle θ that OP makes with the x-axis. (The x-axis being the line determined by $\mathbf{i}$.)

We have

$$x = r\cos\theta \qquad \text{(hence } \tan\theta = \frac{y}{x}\text{)}$$
$$y = r\sin\theta$$

and so

$$\begin{aligned}\mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j}\\ &= r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}\\ &= r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\end{aligned}$$

This equation is called the **polar representation** of $\mathbf{r}$.

A vector of length 1 is called a **unit vector**. The unit vector in the direction of $\mathbf{r}$ is $\hat{\mathbf{r}} = \frac{1}{r}\mathbf{r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$.

Examples

1) If P is $(-2, 3)$, then $r = \sqrt{13}$ and $\cos\theta = -\frac{2}{\sqrt{13}}$. Using the fact (draw a diagram!) that P is in the second quadrant and resorting to a calculator we find that $\theta = 2.1588$.

2) Show that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

We shall express $\mathbf{a}$ and $\mathbf{b}$ in Cartesian coordinates. That is, put $\overrightarrow{AB} = a\,\mathbf{i} + b\,\mathbf{j}$ and $\overrightarrow{BC} = c\,\mathbf{i} + d\,\mathbf{j}$. Then $\overrightarrow{AC} = (a + c)\,\mathbf{i} + (b + d)\,\mathbf{j}$.

Suppose that $|\overrightarrow{AC}| > |\overrightarrow{AB}| + |\overrightarrow{BC}|$ and show that this leads to a contradiction. The details will be given in next week's lecture.