MATH1902 Linear Algebra

Lecture 3
Week 2, Semester 1, 2001

5 March, 2001

Lecture Notes: *Vectors* by C. J. Durrant
Available from Kopystop
(36 Mountain Street, Broadway)

Lecturer: Associate Professor D. E. Taylor
Room: 711, Carslaw Building
Office Hour: Tuesday 1pm – 2pm

Enquiries to: First Year Mathematics Office,
5th floor, Carslaw Building

Web:
`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

Objectives

- use the technique of "proof by contradiction" to establish theorems about vectors; for example, the triangle inequality for vectors in two dimensions.

- calculate with Cartesian coordinates in three dimensions.

- draw diagrams of points and vectors in three dimensions.

- know the definition of the length and the direction cosines of a vector in three dimensions.

- know the difference between the coordinates of a point and the coordinates of a vector and use coordinates to calculate the effect of applying a given vector to a given point.

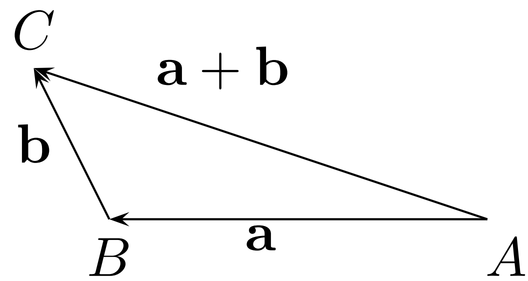
The triangle inequality

At the end of last lecture we wanted to show that for any vectors $\mathbf{a}$ and $\mathbf{b}$ we have

$$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|.$$

This is known as the **triangle inequality** because of the following picture.

$$|\overrightarrow{AB}| = \sqrt{a_1^2 + a_2^2}$$



We shall express $\mathbf{a}$ and $\mathbf{b}$ in Cartesian coordinates. That is, put $\overrightarrow{AB} = a_1\,\mathbf{i} + a_2\,\mathbf{j}$ and $\overrightarrow{BC} = b_1\,\mathbf{i} + b_2\,\mathbf{j}$. Then $\overrightarrow{AC} = (a_1+b_1)\,\mathbf{i} + (a_2+b_2)\,\mathbf{j}$.

Suppose that $|\overrightarrow{AC}| > |\overrightarrow{AB}| + |\overrightarrow{BC}|$. We show that this leads to a contradiction.

Squaring both sides and using the expression for the length of a vector in Cartesian coordinates we have:

$$|\overrightarrow{AC}|^2 > |\overrightarrow{AB}|^2 + 2|\overrightarrow{AB}|\,|\overrightarrow{BC}| + |\overrightarrow{BC}|^2$$

and hence

$$(a_1+b_1)^2+(a_2+b_2)^2$$
$$> a_1^2+a_2^2+2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}+b_1^2+b_2^2.$$

On expanding both sides, cancelling like terms and dividing by 2, this simplifies to

$$a_1b_1+a_2b_2 > \sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}.$$

After squaring again we have

$$a_1^2b_1^2+2a_1a_2b_1b_2+a_2^2b_2^2 > a_1^2b_1^2+a_1^2b_2^2+a_2^2b_1^2+a_2^2b_2^2.$$

Some further cancelling and shifting of $2a_1a_2b_1b_2$ to the right hand side we find that

$$0 > a_1^2b_2^2+a_2^2b_1^2-2a_1a_2b_1b_2=(a_1b_2-a_2b_1)^2.$$

But this cannot happen because the square of a real number is never negative.

Thus we have shown that $|\overrightarrow{AC}| > |\overrightarrow{AB}|+|\overrightarrow{BC}|$ leads to a contradiction and therefore $|\overrightarrow{AC}| \le |\overrightarrow{AB}|+|\overrightarrow{BC}|$. That is, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$.

Coordinates in three dimensions

Vectors book, section 2.5

Choose a point O in space and call it the origin, let $\mathbf{i}$ be a unit vector and let $\mathbf{j}$ be a unit vector that is perpendicular to $\mathbf{i}$. Finally, let $\mathbf{k}$ be a unit vector perpendicular to both $\mathbf{i}$ and $\mathbf{j}$. In fact there are two possibilities for $\mathbf{k}$: one is the negative of the other.

We choose $\mathbf{k}$ as follows. If we grasp the line perpendicular to $\mathbf{i}$ and $\mathbf{j}$ with our right hand so that our fingers curl from $\mathbf{i}$ to $\mathbf{j}$, then our thumb points in the direction of $\mathbf{k}$. This is called a **right-handed** set of unit vectors.

The line through $\mathbf{i}$ is usually called the x-axis.

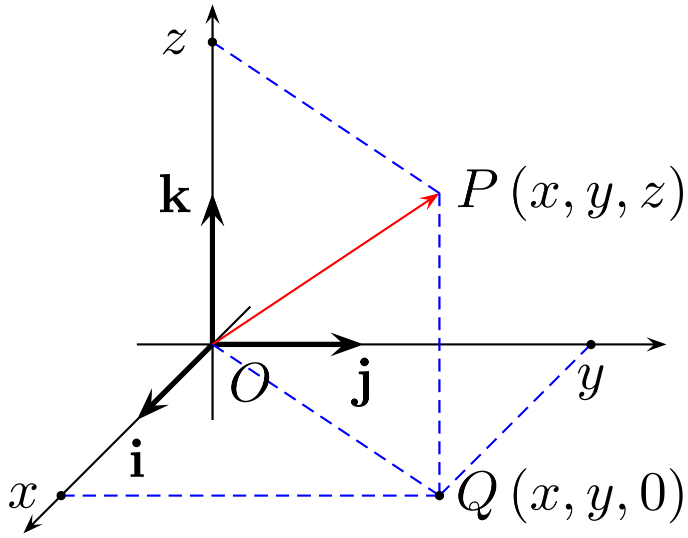
The line through $\mathbf{j}$ is usually called the y-axis.

The line through $\mathbf{k}$ is usually called the z-axis.

The plane containing $\mathbf{i}$ and $\mathbf{j}$ is called the xy-plane.

The plane containing $\mathbf{j}$ and $\mathbf{k}$ is called the yz-plane.

The plane containing $\mathbf{i}$ and $\mathbf{k}$ is called the xz-plane.



Given a point P, with coordinates (x, y, z) relative to the above coordinate system, let $\mathbf{r}$ be the position vector of P with respect to O. If Q is the point with coordinates $(x, y, 0)$, then from the two-dimensional case we have $\overrightarrow{OQ} = x\mathbf{i} + y\mathbf{j}$ and by the triangle rule we have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

Furthermore, by Pythagoras' Theorem and the two-dimensional case we have $|\mathbf{r}|^2 = x^2 + y^2 + z^2$. We often write r for the magnitude of $\mathbf{r}$. That is,

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Note that the axes are right-handed if, when you are standing at O on the xy-plane with z up and y straight ahead, then x is to your right.

Vector algebra in three dimensions

Exactly the same rules apply as before, including the triangle and parallelogram rules for addition as well as all the axioms listed in last lecture, such as $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

These rules allow us to carry out calculations in Cartesian coordinates such as

$$(2\mathbf{i}-3\mathbf{j}+\mathbf{k})+2(\mathbf{i}-4\mathbf{j})=4\mathbf{i}-11\mathbf{j}+\mathbf{k}.$$

If $\overrightarrow{OP_1}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and if $\overrightarrow{OP_2}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$, then the "head minus tail" rule in Cartesian coordinates is

$$\begin{aligned}\overrightarrow{P_1P_2}&=\overrightarrow{OP_2}-\overrightarrow{OP_1}\\&=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}\end{aligned}$$

Angles and direction cosines

Let α be the angle that $\overrightarrow{OP}$ makes with the positive x-axis, let β be the angle that $\overrightarrow{OP}$ makes with the positive y-axis, and let γ be the angle that $\overrightarrow{OP}$ makes with the positive z-axis. Then

$$x = r\cos\alpha, \qquad y = r\cos\beta, \quad \text{and} \quad z = \cos\gamma.$$

We call $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ the **direction cosines** of $\overrightarrow{OP}$.

We have

$$\begin{aligned}\mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\ &= r\cos\alpha\,\mathbf{i} + r\cos\beta\,\mathbf{j} + r\cos\gamma\,\mathbf{k} \\ &= r(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})\end{aligned}$$

The unit vector in the direction of $\mathbf{r}$ is

$$\hat{\mathbf{r}} = \frac{1}{r}\mathbf{r} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

It follows that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

The coordinates of a vector

Suppose that we have chosen an origin of coordinates O and a right-handed coordinate system with mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Given a vector $\mathbf{r}$ we know that there are numbers x, y and z such that $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. We call the triple (x, y, z) the **coordinates** of the vector $\mathbf{r}$. The diagram we drew previously shows that (x, y, z) are the coordinates of the point P such that $\mathbf{r} = \overrightarrow{OP}$.

If A is a point with coordinates (a, b, c), what are the coordinates of the point B such that $\mathbf{r} = \overrightarrow{AB}$?

We have $\overrightarrow{OA} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ and from the triangle rule we have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \mathbf{r}$. That is,

$$\begin{aligned}\overrightarrow{OB} &= (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) + (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \\ &= (a + x)\,\mathbf{i} + (b + y)\,\mathbf{j} + (c + z)\,\mathbf{k}\end{aligned}$$

and therefore the coordinates of B are $(a + x, b + y, c + z)$.

This means that the "action" of a vector on a point is obtained by simply adding their coordinate triples.

