MATH1902 Linear Algebra

Lecture 4
Week 2, Semester 1, 2001

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Lecture Notes: *Vectors* by C. J. Durrant
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Objectives

- be able to calculate the **length** and the **direction cosines** of a vector in three dimensions.
- be able to use vectors to solve **geometry problems** in three dimensions.
- be able to apply the concept of **linear independence** to collections of vectors in two and three dimensions.
- be able to calculate the **scalar product** and the **angle** between two vectors.

Some conventions

In the *Vectors* book (see p. 15) and in the tutorial exercises the following conventions are often used:

$\mathbf{a}$ denotes the position vector $\overrightarrow{OA}$.

a denotes the **magnitude** of $\mathbf{a}$; i.e., $a = |\mathbf{a}|$.

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Example 1

Direction cosines

Given points $P\,(1,-2,5)$ *and* $Q\,(3,1,-1)$*,*

(i) Determine the direction cosines of $\overrightarrow{PQ}$*.*

(ii) Find a point R *such that* $|\overrightarrow{RP}| = 21$ *and with the direction of* $\overrightarrow{RP}$ *opposite to the direction of* $\overrightarrow{PQ}$*.*

We have $\overrightarrow{OP} = \mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k}$ and $\overrightarrow{OQ} = 3\,\mathbf{i} + \mathbf{j} - \mathbf{k}$ so that

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (3\,\mathbf{i} + \mathbf{j} - \mathbf{k}) - (\mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k}) \\ &= 2\,\mathbf{i} + 3\,\mathbf{j} - 6\,\mathbf{k}.\end{aligned}$$

It follows that

$$|\overrightarrow{PQ}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7.$$

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Thus the unit vector in the direction of $\overrightarrow{PQ}$ is $\frac{1}{7}(2\,\mathbf{i}+3\,\mathbf{j}-6\,\mathbf{k})$ and therefore, the direction cosines of $\overrightarrow{PQ}$ are $\boxed{\frac{2}{7}}$, $\boxed{\frac{3}{7}}$ and $\boxed{-\frac{6}{7}}$.

For (ii) we first note that the unit vector in the opposite direction to $\overrightarrow{PQ}$ is $-\frac{1}{7}(2\,\mathbf{i}+3\,\mathbf{j}-6\,\mathbf{k})$ and then we multiply by 21 to get a vector of length 21. This means we want to find the point R such that $\overrightarrow{RP}=-3(2\,\mathbf{i}+3\,\mathbf{j}-6\,\mathbf{k})$. To find the coordinates of R we need to calculate $\overrightarrow{OR}$. That is,

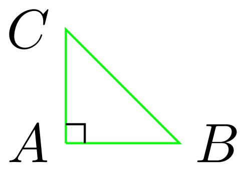
$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OP}+\overrightarrow{PR}\\ &= \overrightarrow{OP}-\overrightarrow{RP}\\ &= (\mathbf{i}-2\,\mathbf{j}+5\,\mathbf{k})-(-3)(2\,\mathbf{i}+3\,\mathbf{j}-6\,\mathbf{k})\\ &= 7\,\mathbf{i}+7\,\mathbf{j}-13\,\mathbf{k}.\end{aligned}$$

Thus the point R has coordinates $(7,7,-13)$.

Remember that the coordinates of the **position vector** $\overrightarrow{OR}$ are the coordinates of the **point** R.

Example 2

Triangles in three dimensions



Find the lengths of the sides of the triangle whose vertices are $A\,(2,4,-1)$, $B\,(4,5,1)$, $C\,(3,6,-3)$ and show that the triangle is right-angled.

We have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} =$ $(4\,\mathbf{i} + 5\,\mathbf{j} + \ \mathbf{k}) - (2\,\mathbf{i} + 4\,\mathbf{j} - \ \mathbf{k}) = 2\,\mathbf{i} + \ \mathbf{j} + 2\,\mathbf{k}$ and so $|\overrightarrow{AB}| = \sqrt{2^2 + 1 + 2^2} = 3$.

Similarly, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} =$ $(3\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}) - (4\,\mathbf{i} + 5\,\mathbf{j} + \ \mathbf{k}) = -\,\mathbf{i} + \ \mathbf{j} - 4\,\mathbf{k}$ and so $|\overrightarrow{BC}| = \sqrt{1 + 1 + 16} = \sqrt{18}$.

And again, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} =$ $(2\,\mathbf{i} + 4\,\mathbf{j} - \ \mathbf{k}) - (3\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}) = -\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$ and so $|\overrightarrow{CA}| = \sqrt{1 + 2^2 + 2^2} = 3$.

Finally, we observe that $3^2 + 3^2 = \sqrt{18}^{\,2}$ and so the triangle is right-angled.

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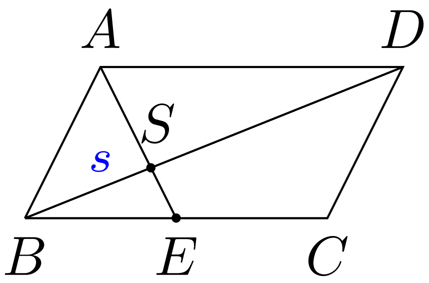
Example 3

A theorem about parallelograms

Given a parallelogram $ABCD$ where the line joining A to the mid-point E of side BC meets the diagonal BD at S, prove that $\overrightarrow{BS} = \frac{1}{3}\overrightarrow{BD}$.

(A parallelogram must lie in a plane but the following calculations do not depend on knowing where it is located.)



For any point O in space we have $\overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC})$, because E is the mid-point of BC.

Also, because $ABCD$ is a parallelogram, we have $\overrightarrow{BC} = \overrightarrow{AD}$ and therefore $\overrightarrow{OC} - \overrightarrow{OB} = \overrightarrow{OD} - \overrightarrow{OA}$.

Hence we can express $\overrightarrow{OD}$ in terms of the other vectors: $\overrightarrow{OD} = \overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}$.

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Furthermore for some scalar s we have

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OB} + s\overrightarrow{BD}\\ &= \overrightarrow{OB} + s(\overrightarrow{OD} - \overrightarrow{OB})\\ &= \overrightarrow{OB} + s(\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}) - s\overrightarrow{OB}\\ &= (1-2s)\overrightarrow{OB} + s\overrightarrow{OA} + s\overrightarrow{OC}\end{aligned}$$

Similarly, for some scalar t we have

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OA} + t\overrightarrow{AE}\\ &= \overrightarrow{OA} + t(\overrightarrow{OE} - \overrightarrow{OA})\\ &= \overrightarrow{OA} + \frac{t}{2}\overrightarrow{OB} + \frac{t}{2}\overrightarrow{OC} - t\overrightarrow{OA}\\ &= (1-t)\overrightarrow{OA} + \frac{t}{2}\overrightarrow{OB} + \frac{t}{2}\overrightarrow{OC}\end{aligned}$$

Equating these two expressions for $\overrightarrow{OS}$ and setting $O = B$ we find that

$$\boxed{s\overrightarrow{BA} + s\overrightarrow{BC} = (1-t)\overrightarrow{BA} + \frac{t}{2}\overrightarrow{BC}.}$$

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That is,

$$(s+t-1)\overrightarrow{BA} = (\frac{t}{2} - s)\overrightarrow{BC}$$

and since these vectors have different directions, we have

$$s+t-1=0 \qquad \text{and} \qquad \frac{t}{2} - s = 0.$$

We see directly from these equations that $s = \frac{1}{3}$. That is, $\overrightarrow{BS} = \overrightarrow{OS} - \overrightarrow{OB} = \frac{1}{3}\overrightarrow{BD}$, as required.

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Linear independence

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be **linearly independent** if the only solution to the equation $\lambda\,\mathbf{a} + \mu\,\mathbf{b} = 0$ is $\lambda = \mu = 0$.

In fact $\mathbf{a}$ and $\mathbf{b}$ are linearly independent if they are both non-zero and have different directions.

An expression of the form $\lambda\,\mathbf{a} + \mu\,\mathbf{b}$ is called a **linear combination** of $\mathbf{a}$ and $\mathbf{b}$.

Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are said to be **linearly independent** if the only solution to the equation $\lambda\,\mathbf{a} + \mu\,\mathbf{b} + \nu\,\mathbf{c} = 0$ is $\lambda = \mu = \nu = 0$.

For example, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are linearly independent.

In the plane there are no three vectors that are linearly independent. This means that one of the vectors can be expressed as a linear combination of the other two.

Some Greek letters

α	alpha	ε	epsilon
β	beta	λ	lambda
γ	gamma	μ	mu
δ	delta	ν	nu

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Scalar products

Vectors book: Chapter 3, section 3.2

Given vectors $\mathbf{u}$ and $\mathbf{v}$, choose a point O and let P and Q be the points such that $\overrightarrow{OP} = \mathbf{u}$ and $\overrightarrow{OQ} = \mathbf{v}$.

The angle $\theta = \angle POQ$ such that $0 \leq \theta \leq \pi$ is called the **angle between** $\mathbf{u}$ and $\mathbf{v}$.

The **scalar product** or **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta}$$

Note: If θ is **acute** (i.e., $0 \leq \theta < \pi/2$), then $\cos\theta > 0$ and so $\mathbf{u} \cdot \mathbf{v} > 0$.

If θ is **obtuse** (i.e., $\pi/2 < \theta \leq \pi$), then $\cos\theta < 0$ and so $\mathbf{u} \cdot \mathbf{v} < 0$.

If $\theta = \pi/2$, then $\cos\theta = 0$ and so $\mathbf{u} \cdot \mathbf{v} = 0$.

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