MATH1902 Linear Algebra

Lecture 5
Week 3, Semester 1, 2001

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Lecture Notes: *Vectors* by C. J. Durrant
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Objectives

- be able to calculate the scalar product of vectors using either lengths and angles or Cartesian coordinates
- know the algebraic laws satisfied by the scalar product
- be able to draw the projection of a vector in a given direction and calculate the projection using scalar products.

The scalar product

Recall from the last lecture that if θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then the **scalar product** or **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta}$$

The scalar product of vectors has the following properties:

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. For all scalars s,
   - $(s\mathbf{u}) \cdot \mathbf{v} = s(\mathbf{u} \cdot \mathbf{v})$
   - $\mathbf{u} \cdot (s\mathbf{v}) = s(\mathbf{u} \cdot \mathbf{v})$
3. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$,
   - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
   - $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. for all $\mathbf{u}$, we have $\mathbf{u} \cdot \mathbf{u} \geq 0$.
5. $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Examples

Vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **perpendicular** or **orthogonal** if the angle between them is $\pi/2$. This is equivalent to $\mathbf{u} \cdot \mathbf{v} = 0$.

Examples: We have $\mathbf{i} \cdot \mathbf{j} = 0$, $\mathbf{i} \cdot \mathbf{k} = 0$ and $\mathbf{j} \cdot \mathbf{k} = 0$.

For any vector $\mathbf{u}$ we have $\boxed{\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2}$.

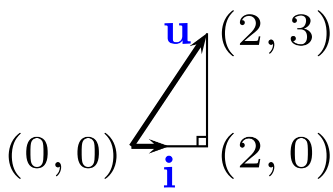
Examples: We have $\mathbf{i} \cdot \mathbf{i} = 1$, $\mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{k} \cdot \mathbf{k} = 1$.

Because of the above properties we call $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ an **orthonormal** basis for $\mathbb{R}^3$.

What is the scalar product of $\mathbf{u} = 2\,\mathbf{i} + 3\,\mathbf{j}$ *and* $\mathbf{i}$*?*

We have $|\mathbf{u}| = \sqrt{13}$ and we can see directly from the diagram that the cosine of the angle between the vectors is $\dfrac{2}{\sqrt{13}}$. Thus

$$\mathbf{u} \cdot \mathbf{i} = \sqrt{13} \times 1 \times \frac{2}{\sqrt{13}} = 2.$$



Direction cosines and the scalar product

In previous lectures we saw that the direction cosines of a vector $\mathbf{r}$ are the cosines of the angles that $\mathbf{r}$ makes with the x-, y- and z-axes. Furthermore, if these angles are α, β and γ, and if $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then

$$x = r\cos\alpha, \qquad y = r\cos\beta, \quad \text{and} \quad z = \cos\gamma,$$

where $r = |\mathbf{r}|$ is the length of $\mathbf{r}$.

From the definition of the scalar product we have

$$\begin{aligned} \mathbf{r}\cdot\,\mathbf{i} &= r\cos\alpha = x, \\ \mathbf{r}\cdot\,\mathbf{j} &= r\cos\beta = y, \quad \text{and} \\ \mathbf{r}\cdot\,\mathbf{k} &= r\cos\gamma = z \end{aligned}$$

Thus

$$\mathbf{r} = (\mathbf{r}\cdot\,\mathbf{i})\,\mathbf{i} + (\mathbf{r}\cdot\,\mathbf{j})\,\mathbf{j} + (\mathbf{r}\cdot\,\mathbf{k})\,\mathbf{k}.$$

The distributive law for the scalar product

All the properties of the scalar product that we have listed, except for the **distributive law**, can be seen directly from the definition. We shall now prove the distributive law.

Applying the formula on the previous slide to the vectors $\mathbf{v}$ and $\mathbf{v}$ we have

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{i})\,\mathbf{i} + (\mathbf{u} \cdot \mathbf{j})\,\mathbf{j} + (\mathbf{u} \cdot \mathbf{k})\,\mathbf{k}$$
$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\,\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\,\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\,\mathbf{k}$$

and therefore

$$\begin{aligned}\mathbf{u} + \mathbf{v} = ((\mathbf{u} \cdot \mathbf{i}) + (\mathbf{v} \cdot \mathbf{i}))\,\mathbf{i} \\ + ((\mathbf{u} \cdot \mathbf{j}) + (\mathbf{v} \cdot \mathbf{j}))\,\mathbf{j} \\ + ((\mathbf{u} \cdot \mathbf{k}) + (\mathbf{v} \cdot \mathbf{k}))\,\mathbf{k}\end{aligned}$$

On the other hand we have

$$\mathbf{u} + \mathbf{v} =$$
$$((\mathbf{u} + \mathbf{v}) \cdot \mathbf{i})\,\mathbf{i} + ((\mathbf{u} + \mathbf{v}) \cdot \mathbf{j})\,\mathbf{j} + ((\mathbf{u} + \mathbf{v}) \cdot \mathbf{k})\,\mathbf{k}$$

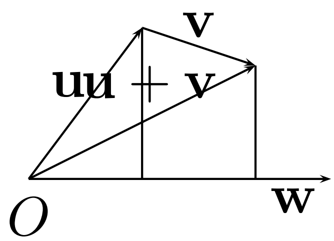
and on comparing the coefficients of $\mathbf{i}$ we see that

$$(\mathbf{u}+\mathbf{v})\cdot\ \mathbf{i}=\mathbf{u}\cdot\ \mathbf{i}+\mathbf{v}\cdot\ \mathbf{i}$$

In fact, given **any** three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can choose our coordinate system so that $\mathbf{i}$ is the unit vector in the direction of $\mathbf{w}$ and then multiplying the last equation by $|\mathbf{w}|$ we find that

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}.$$

This is the **distributive law**.



Some examples:

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=|\mathbf{u}|^2+2\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2$$
$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=|\mathbf{u}|^2-|\mathbf{v}|^2.$$

The Cartesian form of the scalar product

Given vectors $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$, $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and using the distributive laws we have

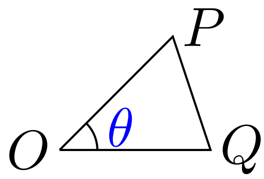
$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k})\cdot(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k})\\
&= u_1v_1\,\mathbf{i}\cdot\,\mathbf{i} + u_1v_2\,\mathbf{i}\cdot\,\mathbf{j} + u_1v_3\,\mathbf{i}\cdot\,\mathbf{k}\\
&+ u_2v_1\,\mathbf{j}\cdot\,\mathbf{i} + u_2v_2\,\mathbf{j}\cdot\,\mathbf{j} + u_2v_3\,\mathbf{j}\cdot\,\mathbf{k}\\
&+ u_3v_1\,\mathbf{k}\cdot\,\mathbf{i} + u_3v_2\,\mathbf{k}\cdot\,\mathbf{j} + u_3v_3\,\mathbf{k}\cdot\,\mathbf{k}\\
&= u_1v_1 + u_2v_2 + u_3v_3.
\end{aligned}$$

That is

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3}\,.$$

The cosine rule for a triangle

Given a triangle OPQ



with angle $\theta = \angle POQ$, we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ and therefore

$$\begin{aligned}
|\overrightarrow{PQ}|^2 &= \overrightarrow{PQ} \cdot \overrightarrow{PQ} \\
&= (\overrightarrow{OQ} - \overrightarrow{OP}) \cdot (\overrightarrow{OQ} - \overrightarrow{OP}) \\
&= |\overrightarrow{OQ}|^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OQ} + |\overrightarrow{OP}|^2 \\
&= |\overrightarrow{OQ}|^2 - 2|\overrightarrow{OP}|\,|\overrightarrow{OQ}| \cos\theta + |\overrightarrow{OP}|^2.
\end{aligned}$$

We can write this as

$$\cos\theta = \frac{|\overrightarrow{OQ}|^2 + |\overrightarrow{OP}|^2 - |\overrightarrow{PQ}|^2}{2|\overrightarrow{OP}|\,|\overrightarrow{OQ}|}.$$

This formula is known as the **cosine rule**.

Projection of a vector

The scalars x, y and z in the expression $\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ are called the **components** of $\mathbf{v}$ in the directions $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

In general, if $\mathbf{u}$ is any non-zero vector, we define the **component of $\mathbf{v}$ in the direction of $\mathbf{u}$** to be $|\mathbf{v}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

That is, the component of $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|} = \mathbf{v}\cdot\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) = \mathbf{v}\cdot\hat{\mathbf{u}}$$

where $\hat{\mathbf{u}}$ is the unit vector in the direction of $\mathbf{u}$.

The vector $(\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ is called the **projection** of $\mathbf{v}$ in the direction of $\mathbf{u}$. Note that

$$(\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = (\mathbf{v}\cdot\frac{\mathbf{u}}{|\mathbf{u}|})\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}.$$