

MATH1902 Linear Algebra

Lecture 6

Week 3, Semester 1, 2001

13 March, 2001

Lecture Notes: *Vectors* by C. J. Durrant
Available from Kopystop
(36 Mountain Street, Broadway)

Lecturer: Associate Professor D. E. Taylor
Room: 711, Carlaw Building
Office Hour: Tuesday 1pm – 2pm

Enquiries to: First Year Mathematics Office,
5th floor, Carlaw Building

Web:

www.maths.usyd.edu.au/u/UG/JM/MATH1902/

Objectives

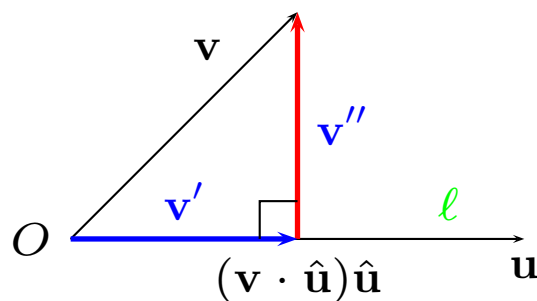
- be able to calculate the **resolution** of a vector into components parallel and perpendicular to a given vector.
- be able to calculate the **vector product** of two vectors
- know the algebraic rules satisfied by vector products.

Resolution of a vector with respect to a line

Given a line ℓ and a vector \mathbf{v} we want to write \mathbf{v} as the sum of a vector **parallel** to ℓ and a vector **perpendicular** to ℓ .

To do this we first choose a vector $\mathbf{u} \neq \mathbf{0}$ whose direction is the same as ℓ .

In the following diagram, which is almost the same as the one found at the end of the last lecture, resolving \mathbf{v} in the direction \mathbf{u} amounts to writing \mathbf{v} as the sum of the **blue** and the **red** vectors.



If we write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$, where \mathbf{v}' is parallel to \mathbf{u} and \mathbf{v}'' is perpendicular to \mathbf{u} , then \mathbf{v}' is the projection of \mathbf{v} onto \mathbf{u} and so

$$\mathbf{v}' = (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$$

$$\mathbf{v}'' = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$$

Example

Suppose that $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and that $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Find the resolution of \mathbf{v} into components parallel and perpendicular to \mathbf{u} .

In this case we have $|\mathbf{u}|^2 = 36 + 4 + 9 = 49$ and $\mathbf{v} \cdot \mathbf{u} = 4$.

Therefore the component of \mathbf{v} **parallel** to \mathbf{u} is

$$\mathbf{v}' = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \mathbf{u} = \frac{4}{49}(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

and the component **perpendicular** to \mathbf{u} is

$$\begin{aligned} \mathbf{v}'' &= (\mathbf{i} + \mathbf{j}) - \frac{4}{49}(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= \frac{1}{49}(25\mathbf{i} + 57\mathbf{j} - 12\mathbf{k}) \end{aligned}$$

As a check we calculate the scalar product of these vectors. It is

$$\frac{4}{49}(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \frac{1}{49}(25\mathbf{i} + 57\mathbf{j} - 12\mathbf{k}) = 0$$

as required.

Projection of a vector onto a plane

Given vectors $\mathbf{u} \neq \mathbf{0}$ and \mathbf{v} we want to find the *projection* of \mathbf{v} onto the plane perpendicular to \mathbf{u} .

If we write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$, where \mathbf{v}' is parallel to \mathbf{u} and \mathbf{v}'' is perpendicular to \mathbf{u} , then \mathbf{v}'' is the projection of \mathbf{v} onto the plane perpendicular to \mathbf{u} .

That is, the **red** vector in the previous diagram is the projection onto the **plane** perpendicular to \mathbf{u} and the **blue** vector is the projection onto the **line** along \mathbf{u} .

Recall that the formula for \mathbf{v}'' is

$$\mathbf{v}'' = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \mathbf{u}$$

If \mathbf{w} is another vector and if \mathbf{w}'' is its projection onto the plane perpendicular to \mathbf{u} , then the projection of $\mathbf{v} + \mathbf{w}$ onto this plane is $\mathbf{v}'' + \mathbf{w}''$. That is, we may write

$$(\mathbf{v} + \mathbf{w})'' = \mathbf{v}'' + \mathbf{w}''$$

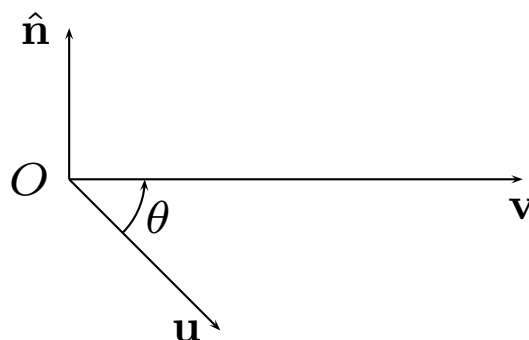
The vector product

Given non-zero vectors \mathbf{u} and \mathbf{v} and given that θ is the angle between them, we define the **vector product** of \mathbf{u} and \mathbf{v} to be the vector

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \sin \theta \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to \mathbf{u} and \mathbf{v} such that \mathbf{u} , \mathbf{v} and $\hat{\mathbf{n}}$ form a right-handed system. (That is, if we curl the fingers of our right hand from \mathbf{u} to \mathbf{v} , then our thumb points along $\hat{\mathbf{n}}$.)

If either \mathbf{u} or \mathbf{v} is $\mathbf{0}$, we define the vector product $\mathbf{u} \times \mathbf{v}$ to be $\mathbf{0}$.



(*Aside.* While the scalar product generalises to spaces of any dimension, the form of vector product given here is somewhat special to three dimensions.)

The algebraic laws of the vector product

The vector product has the following properties:

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. For all scalars s ,
 - $(s\mathbf{u}) \times \mathbf{v} = s(\mathbf{u} \times \mathbf{v})$
 - $\mathbf{u} \times (s\mathbf{v}) = s(\mathbf{u} \times \mathbf{v})$
3. For all vectors \mathbf{u} , \mathbf{v} and \mathbf{w} ,
 - $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
 - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. For all vectors \mathbf{u} and \mathbf{v} ,
 - $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
 - $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
5. $|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2 |\mathbf{v}|^2$.

Note that in general $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

The distributive law for vector products

Just as we found for the scalar product, so too for the vector product: namely, all the properties of the vector product listed on the previous slide, except for the **distributive law**, can be seen directly from the definition.

We shall now prove the distributive law.

First notice that the length of the projection \mathbf{v}'' onto the plane perpendicular to \mathbf{u} is $|\mathbf{v}| \sin \theta$, where θ is the angle between \mathbf{u} and \mathbf{v} . (Check this from the diagram or directly calculate $\mathbf{v}'' \cdot \mathbf{v}''$.)

Since \mathbf{u} and \mathbf{v}'' are perpendicular, we see that $\mathbf{u} \times \mathbf{v}'' = |\mathbf{u}| |\mathbf{v}| \sin \theta \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{u} \times \mathbf{v}$. That is, $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{v}''$.

We have seen previously that if \mathbf{w} is another vector and if \mathbf{w}'' is its projection onto the plane perpendicular to \mathbf{u} , then the projection $(\mathbf{v} + \mathbf{w})''$ of $\mathbf{v} + \mathbf{w}$ onto this plane is $\mathbf{v}'' + \mathbf{w}''$.

Therefore, the distributive law

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ becomes

$$\mathbf{u} \times (\mathbf{v}'' + \mathbf{w}'') = \mathbf{u} \times \mathbf{v}'' + \mathbf{u} \times \mathbf{w}''.$$

But this is now clear geometrically because the vector product of \mathbf{u} with a vector perpendicular to \mathbf{u} is obtained by rotating the vector through $\pi/2$ about \mathbf{u} and then multiplying by $|\mathbf{u}|$.

This completes the proof of the distributive law.

The Cartesian form of the vector product

For any vector \mathbf{v} we have $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ and for the orthonormal basis \mathbf{i} , \mathbf{j} , \mathbf{k} we have

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \quad \text{and} \\ \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}.$$

If we write \mathbf{u} and \mathbf{v} in Cartesian form as

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \quad \text{and}$$

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k},$$

then using the algebraic laws we find that

$$\mathbf{u} \times \mathbf{v} =$$

$$(u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

Example:

$$(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 10\mathbf{i} - 8\mathbf{j} + \mathbf{k}.$$

The determinant notation

We introduce the notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

so that the formula for $\mathbf{u} \times \mathbf{v}$ becomes

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

In fact we often write this formula as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

This is the **determinant** formula for $\mathbf{u} \times \mathbf{v}$. We shall have a lot more to say about determinants later in the course. For now we simply regard it as a useful abbreviation.