MATH1902 Linear Algebra

Lecture 7
Week 4, Semester 1, 2001

19 March, 2001

Lecture Notes: *Vectors* by C. J. Durrant
Available from Kopystop
(36 Mountain Street, Broadway)

Lecturer: Associate Professor D. E. Taylor
Room: 711, Carslaw Building
Office Hour: Tuesday 1pm – 2pm

Enquiries to: First Year Mathematics Office,
5th floor, Carslaw Building

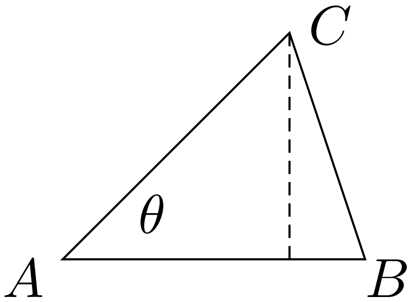
Web:
`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

Objectives

- express the area of a triangle in terms of the vector product
- express the area of a parallelogram in terms of the vector product
- calculate the value of the scalar triple product of three vectors.
- express the volume of a parallelepiped in terms of the scalar triple product.
- know the algebraic properties of the scalar triple product.
- calculate the shortest distance from a point to a line.
- calculate the shortest distance between two lines.

The area of a triangle

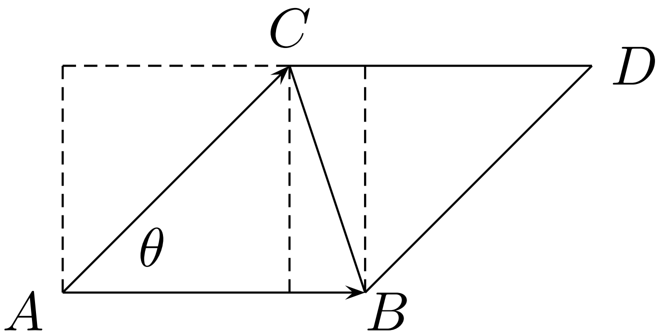
The area of a triangle ABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.



The height of the triangle is $|\overrightarrow{AC}| \sin\theta$, its base has length $|\overrightarrow{AB}|$ and therefore its area is $\frac{1}{2}$ base $\times$ height $= \frac{1}{2}|\overrightarrow{AB}|\,|\overrightarrow{AC}| \sin\theta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

The area of a parallelogram

The area of the parallelogram $ABDC$ shown below is $|\overrightarrow{AB} \times \overrightarrow{AC}|$.



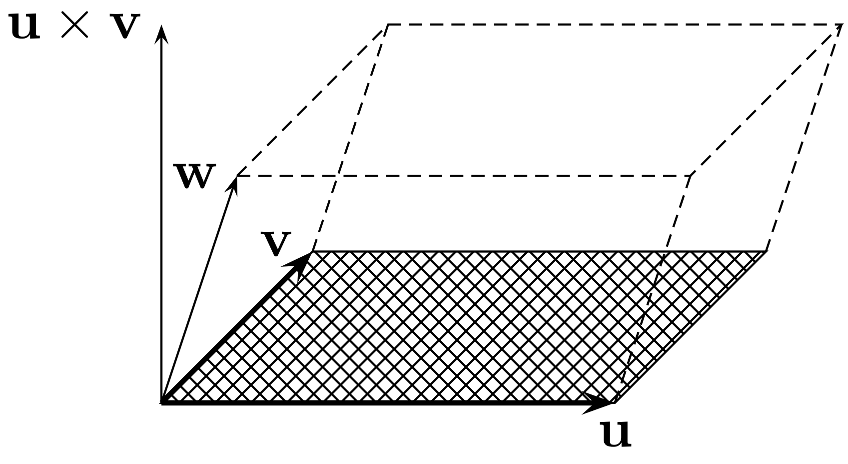
We see from the diagram that area of the parallelogram is the same as the area of the rectangle with the same base and the same height. It is also clear that its area is twice the area of the triangle ABC.

The scalar triple product

If we take the scalar product of $\mathbf{u} \times \mathbf{v}$ with $\mathbf{w}$ we get the **scalar triple product** $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. That is,

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

The absolute value of this quantity is the volume ($=$ base $\times$ height) of the **parallelepiped** with sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.



To see why this is so, note that the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$ and the height of the parallelepiped is $|\mathbf{w}| \cos \theta$, where θ is the angle between $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$.

A formula for the scalar triple product

If we write $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in Cartesian form as

$$\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k},$$
$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k} \quad \text{and}$$
$$\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k},$$

then

$$\mathbf{u} \times \mathbf{v} =$$
$$(u_2v_3 - u_3v_2)\,\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

and therefore

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] =$$
$$(u_2v_3 - u_3v_2)w_1 - (u_1v_3 - u_3v_1)w_2 + (u_1v_2 - u_2v_1)w_3$$

Determinant notation

Recall that in the last lecture we introduced the notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and so the previous formula can be written

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] =$$

$$\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} w_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} w_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} w_3$$

and this can be further abbreviated to

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The formula for $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ on the previous slide shows that we have

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = [\mathbf{v}, \mathbf{w}, \mathbf{u}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Algebraic properties of the scalar triple product

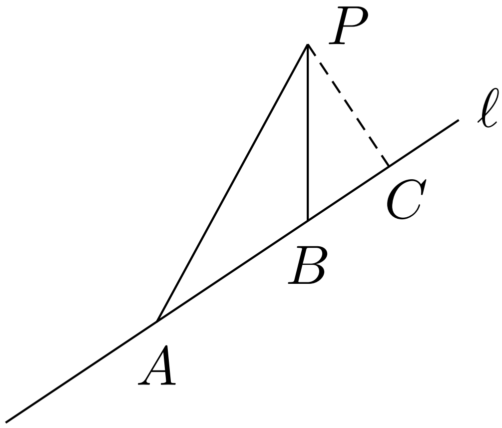
There are twelve expressions that can be made by combining $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ using a scalar and a vector product. This is because there are six permutations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and then two ways to place the dot and the cross.

Up to a sign, all twelve expressions represent the volume of the associated parallelepiped and so all that remains to do is to determine the correct sign. To do this we repeatedly apply the facts that the scalar product is *symmetric* ($\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$), the vector product is *anti-symmetric* ($\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$) and swapping the dot and the cross does not change the sign.

$$
\begin{aligned}
[\mathbf{u}, \mathbf{v}, \mathbf{w}] &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\
&= (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \\
&= (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \\
&= -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \\
&= -(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) \\
&= -(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})
\end{aligned}
$$

The distance of a point from a line

If ℓ is the line through two distinct points A and B and if P is any point in space, what is the shortest distance from P to a point on ℓ?



The shortest distance from P to the line is the length of the perpendicular PC from P.

We see immediately that PC is the altitude of the triangle ABP and from the formula for the area of a triangle we have

$$|\overrightarrow{PC}| = \frac{|\overrightarrow{PA} \times \overrightarrow{PB}|}{|\overrightarrow{AB}|}.$$

The shortest distance between two skew lines

Given a line ℓ through the points A and B and a line m through the points C and D, what is the shortest distance between ℓ and m?

We shall suppose that the lines ℓ and m are **skew**; that is, they are **not** parallel.

Then the vector $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{CD} \neq \mathbf{0}$ is perpendicular to both ℓ and m. If we consider the line h in the direction $\mathbf{n}$ through a point P on ℓ and if we move P along ℓ, then at some point h must cross the line m. That is, we can choose P so that h meets m at a point we shall call Q. Then the length of PQ is the shortest distance between the lines. Up to a sign, the length of PQ is the component of $\overrightarrow{PC}$ in the direction $\mathbf{n}$ (draw a picture!); i.e., $\pm\frac{1}{|\mathbf{n}|}\overrightarrow{PC} \cdot \mathbf{n}$.

We have $\overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{PC}$ and $\overrightarrow{AP} \cdot \mathbf{n} = 0$, therefore $\overrightarrow{AC} \cdot \mathbf{n} = \overrightarrow{PC} \cdot \mathbf{n}$ and so the length of the shortest line segment joining ℓ and m is

$$\frac{|\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD})|}{|\overrightarrow{AB} \times \overrightarrow{CD}|}.$$