

MATH1902 Linear Algebra

Lecture 8

Week 4, Semester 1, 2001

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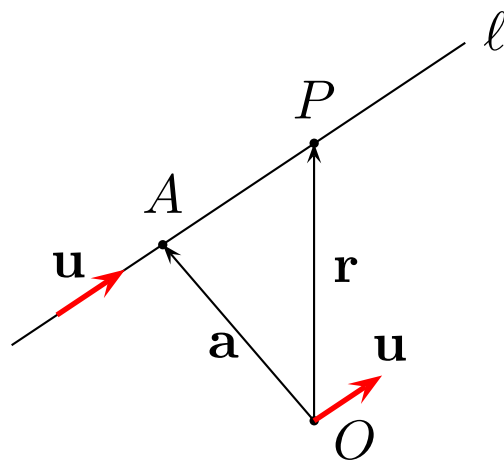
Objectives

- know the vector and coordinate forms of the equation of a line and be able to convert between the two.
- find the equation of the line through two given points.
- be able to express the equation of a line using the cross product.

The vector equation of a line

First choose a point O in 3-dimensional space which we shall use as the origin.

Given a point A and a vector \mathbf{u} , what is the equation of the line through A in the direction \mathbf{u} ?



Let P be any point on the line and let $\mathbf{r} = \overrightarrow{OP}$ be the position vector of P . Then \overrightarrow{AP} is parallel to \mathbf{u} and is therefore equal to $t\mathbf{u}$ for some real number t . (The value of t depends on the position of P .)

Thus, if $\mathbf{a} = \overrightarrow{OA}$ is the position vector of A , then $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ and so

$$\mathbf{r} = \mathbf{a} + t\mathbf{u}.$$

This is the (parametric) **vector equation** of the line.

Example

Find the equation of the line through the point $A(-1, 2, 4)$ in the direction of $\mathbf{u} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Let $P(x, y, z)$ be a point on the line and put $\mathbf{r} = \overrightarrow{OP}$. Then $\mathbf{a} = \overrightarrow{OA} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and

$$\begin{aligned}\mathbf{r} &= \mathbf{a} + t\mathbf{u} \\ &= -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + t(-\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

The position vectors of all the points on the line can be found by varying t . For example, the value $t = 1$ gives $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and so the point $(-2, 3, 5)$ is on the line.

Similarly, the value $t = -4$ gives $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j}$ and thus $(3, -2, 0)$ is on the line.

Coordinate forms of the equation of a line

Given the situation as before with the point A on the line in the direction of \mathbf{b} we may suppose that A has coordinates (a_1, a_2, a_3) and that $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$.

Writing $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ we find that the equation of the line becomes

$$x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} + t(u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}).$$

That is,

$$x = a_1 + tu_1$$

$$y = a_2 + tu_2$$

$$z = a_3 + tu_3$$

and so a single vector equation becomes three scalar equations.

You will often see these equations written in the form

$$\frac{x - a_1}{u_1} = \frac{y - a_2}{u_2} = \frac{z - a_3}{u_3} = t.$$

obtained by eliminating the **parameter** t . These are the **Cartesian** equations of the line.

Example

Determine whether the line through $(1, 1, 1)$ in the direction $\mathbf{i} + 2\mathbf{j}$ intersects the line through $(-1, 2, 3)$ in the direction $\mathbf{i} + \mathbf{k}$.

Suppose that $P(x, y, z)$ is on both lines and put $\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then

$$\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) + s(\mathbf{i} + 2\mathbf{j}) \quad \text{for some } s, \text{ and}$$

$$\mathbf{r} = (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + \mathbf{k}) \quad \text{for some } t.$$

Equating the two expressions for \mathbf{r} and comparing the coefficients of \mathbf{i} , \mathbf{j} and \mathbf{k} we find that

$$1 + s = -1 + t$$

$$1 + 2s = 2$$

$$1 = 3 + t$$

The last two equations give $s = \frac{1}{2}$ and $t = -2$, but then the first equation yields a contradiction. Thus there is no point on both lines. That is, the lines are “skew”, meaning that they do not meet and they are not parallel.

The line through two given points

Find the equation of the line through two given (distinct) points A and B

In this case the direction of the line is $\mathbf{u} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, where $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. Thus the equation of the line is

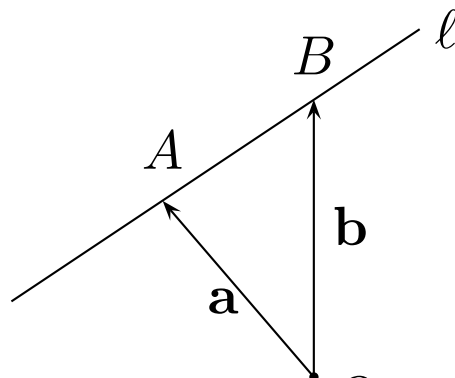
$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

and this can be written

$$\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{b}.$$

If the coordinates of A and B are (a_1, a_2, a_3) and (b_1, b_2, b_3) , the Cartesian form of the equation is

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}.$$



Example

Find the Cartesian form of the equation of the line through the points $(1, 1, 1)$ and $(2, -1, 4)$.

Applying the previous formula immediately leads to the equations

$$\frac{x - 1}{2 - 1} = \frac{y - 1}{-1 - 1} = \frac{z - 1}{4 - 1}.$$

That is,

$$\frac{x - 1}{1} = \frac{y - 1}{-2} = \frac{z - 1}{3}.$$

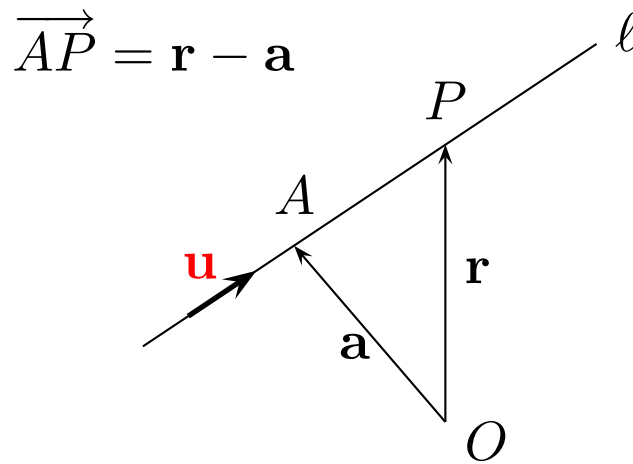
In (scalar) parametric form this becomes

$$x = 1 + t$$

$$y = 1 - 2t$$

$$z = 1 + 3t$$

Vector product form of the equation of a line



Another way to express the fact that the point P with position vector $\mathbf{r} = \overrightarrow{OP}$ is on the line through A in the direction \mathbf{u} is to write

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{u} = \mathbf{0}.$$

This can be interpreted as saying that $\mathbf{r} - \mathbf{a}$ is **parallel** to \mathbf{u} .

Conversely, any point P whose position vector \mathbf{r} satisfies this equation is on the line through A in the direction \mathbf{u} .

The Cartesian form of the vector product equation of a line

Continuing with the notation of the previous slide we put $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$, $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$.

The vector product form of the equation of a line is $(\mathbf{r} - \mathbf{a}) \times \mathbf{u} = \mathbf{0}$ and this can also be written $\mathbf{r} \times \mathbf{u} = \mathbf{a} \times \mathbf{u}$. Now the formula for the vector product shows that

$$\begin{aligned}u_3 y - u_2 z &= u_3 a_2 - u_2 a_3 \\-u_3 x + u_1 z &= -u_3 a_1 + u_1 a_3 \\u_2 x - u_1 y &= u_2 a_1 - u_1 a_2\end{aligned}$$

This is yet another way to describe a line