Objectives

- know the vector and coordinate forms of the equation of a plane and be able to convert between the two.

- find the equation of the plane through a given point and perpendicular to a given vector.

- find the equation of the plane through three given points.

- be able to express the equation of a plane using the scalar product.

- find the distance of a point from a plane.
The vector form of the equation of a plane

Given a plane $P$ and a point $A$ on the plane, there is a \textit{unique} line through $A$ that is perpendicular to every line in the plane.

Conversely, given a point $A$ on a line $\ell$, there is a \textit{unique} plane $P$ through $A$ and perpendicular to $\ell$.

The line $\ell$ is called the \textbf{normal} to the plane.

Our goal is to find the equation of the plane $P$.

Let $\mathbf{n}$ be a vector in the direction of $\ell$. If $O$ is the origin and if $P$ is any point on the plane, then $\overrightarrow{AP}$ is in the plane and therefore perpendicular to $\mathbf{n}$. If $\mathbf{r} = \overrightarrow{OP}$ is the position vector of $P$ and if $\mathbf{a} = \overrightarrow{OA}$ is the position vector of $A$, then $\overrightarrow{AP} = \mathbf{r} - \mathbf{a}$ and so

$$\mathbf{(r - a)} \cdot \mathbf{n} = 0$$

This is the \textbf{vector form} of the equation of the plane through $A$ and perpendicular to $\mathbf{n}$.
There are rather a lot of words to describe the idea of “perpendicular”:

- perpendicular
- orthogonal
- at right angles
- normal
The Cartesian form of the equation of
a plane

As usual, we write $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$.

If $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ and if
$\mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k}$, then
$\mathbf{r} - \mathbf{a} = (x - a_1) \mathbf{i} + (y - a_2) \mathbf{j} + (z - a_3) \mathbf{k}$ and so
$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = (x - a_1)n_1 + (y - a_2)n_2 + (z - a_3)n_3$.

Therefore, the equation of the plane becomes

$$(x - a_1)n_1 + (y - a_2)n_2 + (z - a_3)n_3 = 0.$$  

Another way to write this is

$$n_1x + n_2y + n_3z = d$$

where $d = n_1a_1 + n_2a_2 + n_3a_3$. Notice that the coefficients of $x, y$ and $z$ give the components of the normal vector.
Examples

1) What is the equation of the plane through $(1, 1, -2)$ with normal $3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$?

Using the formulas given above we find that it is

$$3(x - 1) + 2(y - 1) + 5(z + 2) = 0,$$

that is, $3x + 2y + 5z = -5$.

2) Find a vector normal to the plane $x - y + 5z = 17$.

The answer is $\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.
The equation of a plane through three non-collinear points

Given points \( A, B \) and \( C \), not all on a line, what is the equation of the plane containing them?

First observe that \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are parallel to the plane. That is, we can picture them as lying in the plane. These vectors are not parallel to each other because the points \( A, B \) and \( C \) are not collinear. Therefore, the vector product \( \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} \) is not \( \mathbf{0} \).

We know that the vector product of two vectors is perpendicular to both and therefore \( \mathbf{n} \) is perpendicular to the plane. Thus we can find the equation of the plane by using any of the previous formulas applied to \( \mathbf{n} \) and \( A \).

For example, to find the plane through \( A (-1, 1, 1) \), \( B (2, 1, 1) \) and \( C (0, 0, -2) \) we first find \( \overrightarrow{AB} = 3 \mathbf{i} \) and \( \overrightarrow{AC} = \mathbf{i} - \mathbf{j} - 3 \mathbf{k} \). Then \( \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = 9 \mathbf{j} - 3 \mathbf{k} \) is normal the plane and so the equation of the plane is

\[
0(x + 1) + 9(y - 1) - 3(z - 1) = 0,
\]

that is, \( 3y - z = 2 \).
The line of intersection of two planes

Find the parametric equation of the line of intersection of the planes \(2x - 3y + 5z = 4\) and \(x + y + z = 7\).

The normal to the first plane is \(\mathbf{n}_1 = 2 \mathbf{i} - 3 \mathbf{j} + 5 \mathbf{k}\) and the normal to the second plane is \(\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}\).

The line of intersection is in both planes and therefore it is perpendicular to both normals. That is, its direction is \(\mathbf{n}_1 \times \mathbf{n}_2 = -8 \mathbf{i} + 3 \mathbf{j} + 5 \mathbf{k}\).

The next step is to find a point on the line of intersection. That is we must solve the simultaneous equations

\[
\begin{align*}
2x - 3y + 5z &= 4 \\
x + y + z &= 7
\end{align*}
\]

for \(x, y\) and \(z\).
To simplify things we can try to find a point where $z = 0$. Then the equations become $2x - 3y = 4$ and $x + y = 7$ and it is not too hard to see that the solution is $x = 5$ and $y = 2$.

Thus the line of intersection goes through the point $(5, 2, 0)$ in the direction $-8\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$. Therefore, its equation is

\[
\begin{align*}
x &= 5 - 8t \\
y &= 2 + 3t \\
z &= 5t
\end{align*}
\]
The parametric form of the equation of a plane

Given an origin $O$, a point $A$, and vectors $b$ and $c$ the position vector $r$ on the plane through $A$ and parallel to $b$ and $c$ can be written

$$ r = a + sb + tc, $$

where $a = \overrightarrow{OA}$. This is the parametric equation of the plane.

If we put $n = b \times c$, then $n$ is normal to the plane and so its equation can be written $(r - a) \cdot n = 0$.

Given points $E$, $F$ and $G$ with position vectors $e$, $f$ and $g$, the plane through $E$, $F$ and $G$ contains the vectors $b = \overrightarrow{EF} = f - e$ and $c = \overrightarrow{EG} = g - e$. Thus the equation of the plane is

$$ r = e + s(f - e) + t(g - e) $$

and this can be written

$$ r = (1 - s - t)e + sf + tg. $$
The distance of a point from a plane

The vector form of the equation to a plane is \((\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0\) and this can be written \(\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}\). Dividing this equation by the length of \(\mathbf{n}\) we find that the equation becomes

\[
\mathbf{r} \cdot \mathbf{\hat{n}} = d,
\]

where \(d = \mathbf{a} \cdot \mathbf{\hat{n}}\) and where \(\mathbf{\hat{n}} = \frac{1}{|\mathbf{n}|} \mathbf{n}\) is a unit vector. By multiplying through by \(-1\), if necessary, we may suppose that \(d \geq 0\).

If \(\mathbf{r}\) is the position vector of the point \(P\) on the plane, then the component of \(\mathbf{r} = \overrightarrow{OP}\) in the direction of \(\mathbf{\hat{n}}\) is \(\mathbf{r} \cdot \mathbf{\hat{n}} = d\). This means that \(d\) is the distance from the origin to the plane.

If \(P'\) is any point in space and if \(\mathbf{r}'\) is its position vector, then its component in the direction of \(\mathbf{\hat{n}}\) is \(d' = \mathbf{r}' \cdot \mathbf{\hat{n}}\). Thus if \(d' \leq d\), then the distance from \(P'\) to the plane is \(d - d'\).

In general, the distance from \(P'\) to the plane is \(|d - d'|\) and \(P'\) is on the same side of the plane as the origin if and only if \(d - d' \geq 0\).
Example

Find the equation of the plane through $A(3, -2, 1)$ with normal $4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and find the distance of the point $B(2, 4, -3)$ from the plane.

The length of the normal $\mathbf{n} = 4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ is $\sqrt{16 + 49 + 16} = 9$, the position vector of $A$ is $\mathbf{a} = \overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{a} \cdot \mathbf{n} = -6$. Therefore, as unit vector normal to the plane we take $\hat{\mathbf{n}} = -\frac{1}{9}(4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})$ and then the equation of the plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = \frac{2}{3}$.

The position vector of $B$ is $\mathbf{b} = \overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and we have $\overrightarrow{OB} \cdot \hat{\mathbf{n}} = -\frac{16}{3}$. Thus, in the notation of the previous slide, we have $d = \frac{2}{3}$ and $d' = -\frac{16}{3}$.

Since $d' < d$ we know that $B$ is on the same side of the plane as the origin and that the distance from $B$ to the plane is $\frac{2}{3} - (-\frac{16}{3}) = 6$. Alternatively we can calculate the distance directly by projecting $\overrightarrow{BA}$ onto the normal $\hat{\mathbf{n}}$ to get $\overrightarrow{BA} \cdot \hat{\mathbf{n}} = (\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \cdot (\frac{-1}{9}(4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})) = 6$.

Here is the picture, looking at the plane side on.

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