MATH1902 Linear Algebra

Lecture 9
Week 5, Semester 1, 2001

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Lecture Notes: *Vectors* by C. J. Durrant
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Objectives

- know the vector and coordinate forms of the equation of a plane and be able to convert between the two.

- find the equation of the plane through a given point and perpendicular to a given vector.

- find the equation of the plane through three given points.

- be able to express the equation of a plane using the scalar product.

- find the distance of a point from a plane.

The vector form of the equation of a plane

Given a plane $\mathcal{P}$ and a point A on the plane, there is a *unique* line through A that is perpendicular to every line in the plane.

Conversely, given a point A on a line ℓ, there is a *unique* plane $\mathcal{P}$ through A and perpendicular to ℓ.

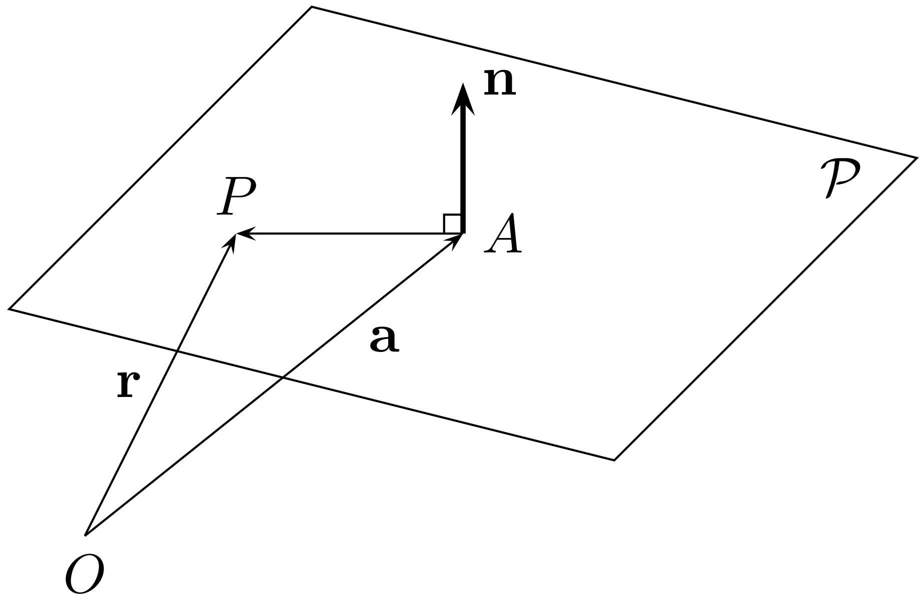
The line ℓ is called the **normal** to the plane.

Our goal is to find the equation of the plane $\mathcal{P}$.

Let $\mathbf{n}$ be a vector in the direction of ℓ. If O is the origin and if P is any point on the plane, then $\overrightarrow{AP}$ is in the plane and therefore perpendicular to $\mathbf{n}$. If $\mathbf{r} = \overrightarrow{OP}$ is the position vector of P and if $\mathbf{a} = \overrightarrow{OA}$ is the position vector of A, then $\overrightarrow{AP} = \mathbf{r} - \mathbf{a}$ and so

$$\boxed{(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0}$$

This is the vector form of the equation of the plane through A and perpendicular to $\mathbf{n}$.



There are rather a lot of words to describe the idea of "perpendicular":

- perpendicular
- orthogonal
- at right angles
- normal

The Cartesian form of the equation of a plane

As usual, we write $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and if
$\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$, then
$\mathbf{r} - \mathbf{a} = (x - a_1)\,\mathbf{i} + (y - a_2)\,\mathbf{j} + (z - a_3)\,\mathbf{k}$ and so
$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = (x - a_1)n_1 + (y - a_2)n_2 + (z - a_3)n_3$.

Therefore, the equation of the plane becomes

$$(x - a_1)n_1 + (y - a_2)n_2 + (z - a_3)n_3 = 0.$$

Another way to write this is

$$\boxed{n_1 x + n_2 y + n_3 z = d}$$

where $d = n_1 a_1 + n_2 a_2 + n_3 a_3$. Notice that the coefficients of x, y and z give the components of the normal vector.

Examples

1) *What is the equation of the plane through* $(1, 1, -2)$ *with normal* $3\,\mathbf{i} + 2\,\mathbf{j} + 5\,\mathbf{k}$*?*

Using the formulas given above we find that it is

$$3(x-1) + 2(y-1) + 5(z+2) = 0,$$

that is, $\boxed{3x + 2y + 5z = -5}$.

2) *Find a vector normal to the plane* $x - y + 5z = 17$*.*

The answer is $\mathbf{i} - \mathbf{j} + 5\,\mathbf{k}$.

The equation of a plane through three non-collinear points

Given points A, B and C, not all on a line, what is the equation of the plane containing them?

First observe that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane. That is, we can picture them as lying in the plane. These vectors are not parallel to each other because the points A, B and C are not collinear. Therefore, the vector product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is not $\mathbf{0}$.

We know that the vector product of two vectors is perpendicular to both and therefore $\mathbf{n}$ is perpendicular to the plane. Thus we can find the equation of the plane by using any of the previous formulas applied to $\mathbf{n}$ and A.

For example, to find the plane through $A\,(-1, 1, 1)$, $B\,(2, 1, 1)$ and $C\,(0, 0, -2)$ we first find $\overrightarrow{AB} = 3\,\mathbf{i}$ and $\overrightarrow{AC} = \mathbf{i} - \mathbf{j} - 3\,\mathbf{k}$. Then $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = 9\,\mathbf{j} - 3\,\mathbf{k}$ is normal the plane and so the equation of the plane is

$$0(x + 1) + 9(y - 1) - 3(z - 1) = 0,$$

that is, $3y - z = 2$.

The line of intersection of two planes

Find the parametric equation of the line of intersection of the planes $2x - 3y + 5z = 4$ *and* $x + y + z = 7$*.*

The normal to the first plane is $\mathbf{n}_1 = 2\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$ and the normal to the second plane is $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

The line of intersection is in both planes and therefore it is perpendicular to both normals. That is, its direction is $\mathbf{n}_1 \times \mathbf{n}_2 = -8\,\mathbf{i} + 3\,\mathbf{j} + 5\,\mathbf{k}$.

The next step is to find a point on the line of intersection. That is we must solve the simultaneous equations

$$\begin{aligned} 2x - 3y + 5z &= 4 \\ x + y + z &= 7 \end{aligned}$$

for x, y and z.

To simplify things we can try to find a point where $z = 0$. Then the equations become $2x - 3y = 4$ and $x + y = 7$ and it is not too hard to see that the solution is $x = 5$ and $y = 2$.

Thus the line of intersection goes through the point $(5, 2, 0)$ in the direction $-8\,\mathbf{i} + 3\,\mathbf{j} + 5\,\mathbf{k}$. Therefore, its equation is

$$x = 5 - 8t$$
$$y = 2 + 3t$$
$$z = 5t$$

The parametric form of the equation of a plane

Given an origin O, a point A, and vectors $\mathbf{b}$ and $\mathbf{c}$ the position vector $\mathbf{r}$ on the plane through A and parallel to $\mathbf{b}$ and $\mathbf{c}$ can be written

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c},$$

where $\mathbf{a} = \overrightarrow{OA}$. This is the *parametric* equation of the plane.

If we put $\mathbf{n} = \mathbf{b} \times \mathbf{c}$, then $\mathbf{n}$ is normal to the plane and so its equation can be written $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

Given points E, F and G with position vectors $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$, the plane through E, F and G contains the vectors $\mathbf{b} = \overrightarrow{EF} = \mathbf{f} - \mathbf{e}$ and $\mathbf{c} = \overrightarrow{EG} = \mathbf{g} - \mathbf{e}$. Thus the equation of the plane is

$$\mathbf{r} = \mathbf{e} + s(\mathbf{f} - \mathbf{e}) + t(\mathbf{g} - \mathbf{e})$$

and this can be written

$$\mathbf{r} = (1 - s - t)\mathbf{e} + s\mathbf{f} + t\mathbf{g}.$$

The distance of a point from a plane

The vector form of the equation to a plane is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ and this can be written $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$. Dividing this equation by the length of $\mathbf{n}$ we find that the equation becomes

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d,$$

where $d = \mathbf{a} \cdot \hat{\mathbf{n}}$ and where $\hat{\mathbf{n}} = \frac{1}{|\mathbf{n}|}\mathbf{n}$ is a unit vector. By multiplying through by -1, if necessary, we may suppose that $d \geq 0$.

If $\mathbf{r}$ is the position vector of the point P on the plane, then the component of $\mathbf{r} = \overrightarrow{OP}$ in the direction of $\hat{\mathbf{n}}$ is $\mathbf{r} \cdot \hat{\mathbf{n}} = d$. This means that d is the distance from the origin to the plane.

If P' is any point in space and if $\mathbf{r}'$ is its position vector, then its component in the direction of $\hat{\mathbf{n}}$ is $d' = \mathbf{r}' \cdot \hat{\mathbf{n}}$. Thus if $d' \leq d$, then the distance from P' to the plane is $d - d'$.

In general, the distance from P' to the plane is $|d - d'|$ and P' is on the same side of the plane as the origin if and only if $d - d' \geq 0$.

Example

Find the equation of the plane through $A\,(3,-2,1)$ *with normal* $4\,\mathbf{i}+7\,\mathbf{j}-4\,\mathbf{k}$ *and find the distance of the point* $B\,(2,4,-3)$ *from the plane.*

The length of the normal $\mathbf{n}=4\,\mathbf{i}+7\,\mathbf{j}-4\,\mathbf{k}$ is $\sqrt{16+49+16}=9$, the position vector of A is $\mathbf{a}=\overrightarrow{OA}=3\,\mathbf{i}-2\,\mathbf{j}+\,\mathbf{k}$ and $\mathbf{a}\cdot\mathbf{n}=-6.$ Therefore, as unit vector normal to the plane we take $\hat{\mathbf{n}}=-\frac{1}{9}(4\,\mathbf{i}+7\,\mathbf{j}-4\,\mathbf{k})$ and then the equation of the plane is $\mathbf{r}\cdot\hat{\mathbf{n}}=\frac{2}{3}$.

The position vector of B is $\mathbf{b}=\overrightarrow{OB}=2\,\mathbf{i}+4\,\mathbf{j}-3\,\mathbf{k}$ and we have $\overrightarrow{OB}\cdot\hat{\mathbf{n}}=-\frac{16}{3}$. Thus, in the notation of the previous slide, we have $d=\frac{2}{3}$ and $d'=-\frac{16}{3}$.

Since $d'<d$ we know that B is on the same side of the plane as the origin and that the distance from B to the plane is $\frac{2}{3}-(-\frac{16}{3})=6.$ Alternatively we can calculate the distance directly by projecting $\overrightarrow{BA}$ onto the normal $\hat{\mathbf{n}}$ to get $\overrightarrow{BA}\cdot\hat{\mathbf{n}}=(\,\mathbf{i}-6\,\mathbf{j}+4\,\mathbf{k})\cdot(\frac{-1}{9}(4\,\mathbf{i}+7\,\mathbf{j}-4\,\mathbf{k}))=6.$

Here is the picture, looking at the plane side on.