

MATH1902 Linear Algebra

Lecture 10
Week 5, Semester 1, 2001

27 March, 2001

Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Understand how finding the intersection of collections of planes and lines is equivalent to solving simultaneous linear equations.
- Use row operations to solve simultaneous linear equations.

Simultaneous linear equations

Definition: A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where x_1, x_2, \dots, x_n are unknowns and a_1, a_2, \dots, a_n and b are constants. No products or powers of the unknowns are allowed. If there are only three unknowns we sometimes denote them by x, y and z rather than x_1, x_2 and x_3 .

For example, $3x - 4y + 2z = -17$ is a linear equation but $x^3 - xy = 7$ is **not** a linear equation.

Recall from the previous lecture that the equation of a plane is a linear equation. The intersection of two planes, which are not parallel, is a line and so a line can be thought of as the solution of a pair of simultaneous linear equations. For example,

$$\begin{aligned}x - 2y + 2z &= -4 \\2x - 3y + 5z &= -1\end{aligned}$$

Solving simultaneous linear equations

To solve the previous equations we can first eliminate x from the second equation by subtracting twice the first equation from it. This produces the pair of equations

$$x - 2y + 2z = -4$$

$$y + z = 7$$

At this point we may introduce a “parameter” t and put $z = t$. Then from the second equation we obtain $y = -t + 7$ and substituting the values of y and z into the first equation gives $x = -4t + 10$. Thus the solution of our original pair of equations is

$$x = -4t + 10$$

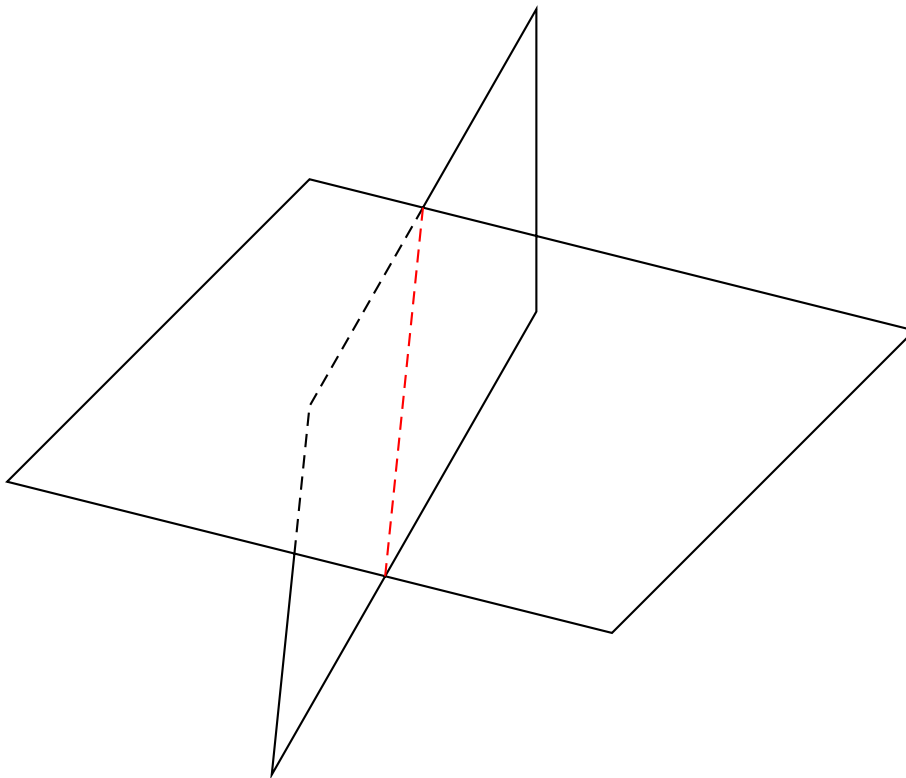
$$y = -t + 7$$

$$z = t$$

This is the parametric equation of the line through the point $(10, 7, 0)$ in the direction $-4\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Connection with vector methods

We can relate this to the previous method of solution by noting that the normal to the first plane is $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, the normal to the second plane is $\mathbf{n}_2 = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and their vector product is $\mathbf{n}_1 \times \mathbf{n}_2 = -4\mathbf{i} - \mathbf{j} + \mathbf{k}$, which is the direction of the line that we found above.



Solving three simultaneous equations

Suppose that in addition to the two equations considered in the previous slides we have a third equation, giving the following system of simultaneous linear equations:

$$x - 2y + 2z = -4$$

$$2x - 3y + 5z = -1$$

$$2x + y + 11z = 33$$

This can be thought of as the intersection of three planes. We found that the intersection of the first two planes was a line and since a line meets a plane in a single point (or else is parallel to the plane) we expect that there will be only **one** point that satisfies all three equations.

Gaussian elimination

To check that the equations have a single solution, we begin as before and use the first equation to eliminate x from the other two. We shall refer to the equations as **rows** and use the symbols R_1 , R_2 and R_3 to refer to the first, second and third equations. Then an expression of the form $R_2 := R_2 - 2R_1$ means that we subtract twice equation 1 from equation 2.

Thus to eliminate x from the second and third equations we perform the operations $R_2 := R_2 - 2R_1$ and $R_3 := R_3 - 2R_1$:

$$x - 2y + 2z = -4$$

$$y + z = 7$$

$$5y + 7z = 41$$

The next step is to use $R_3 := R_3 - 5R_2$ to eliminate y from the third equation:

$$x - 2y + 2z = -4$$

$$y + z = 7$$

$$2z = 6$$

We could now use $R_3 := \frac{1}{2}R_3$ to get $z = 3$.

This process is known as **Gaussian elimination** after Karl Friedrich Gauss, 1777-1855.

In using this method it is somewhat tedious to continually write out the variables. All that really matters are their coefficients and so we omit the variables and simply write out the equations as rows of numbers:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 2 & -3 & 5 & -1 \\ 2 & 1 & 11 & 33 \end{array} \right]$$

This is called the *augmented matrix* of the equations and the array

$$\left[\begin{array}{ccc} 1 & -2 & 2 \\ 2 & -3 & 5 \\ 2 & 1 & 11 \end{array} \right]$$

is called the *coefficient matrix*.

The coefficient matrix has one **row** for each equation and one **column** for each variable. The augmented matrix has an extra column for the right hand sides of the equations; it is usually separated from the coefficient matrix by a vertical line.

Row operations

The solution process (**Gaussian elimination**) that we carried out above can now be written

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 2 & -3 & 5 & -1 \\ 2 & 1 & 11 & 33 \end{array} \right] \xrightarrow{\substack{R_2 := R_2 - 2R_1 \\ R_3 := R_3 - 2R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 0 & 1 & 1 & 7 \\ 0 & 5 & 7 & 41 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 0 & 1 & 1 & 7 \\ 0 & 5 & 7 & 41 \end{array} \right] \xrightarrow{R_3 := R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & 6 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & 6 \end{array} \right] \xrightarrow{R_3 := \frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

The process of replacing one row by a linear combination of other rows is called a **row operation**.

Elementary row operations

The most important point is

*Each augmented matrix obtained throughout this process must represent a system of equations with **exactly** the same solutions as the original equations.*

In order not to lose solutions or to create new ones we must make sure that our row operations are **reversible**.

We do this by restricting ourselves to just three kinds of row operations (called **elementary row operations (ERO)**).

Type 1: $R_i := R_i + \lambda R_j$, where $i \neq j$.

Replace row i by itself plus a multiple of row j .

Type 2: $R_i := \lambda R_i$, where $\lambda \neq 0$.

Replace row i by a non-zero multiple of itself.

Type 3: $R_i \leftrightarrow R_j$

Swap rows i and j .