MATH1902 Linear Algebra

Lecture 11
Week 6, Semester 1, 2001

2 April, 2001

Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Use row operations to bring an augmented matrix to **row echelon form**.
- Use **back substitution** applied to the row echelon form to solve a system of linear equations.
- Recognise consistent and inconsistent systems of equations.

Row echelon form

A matrix is said to be in **row echelon form** if it satisfies the following conditions:

- Any zero rows come after all nonzero rows
- In every nonzero row the leading entry (i.e., the left-most nonzero number) must be a 1.
- The leading entry of a nonzero row occurs further to the right than the leading entry in any previous row.

$$\begin{bmatrix} \textcircled{1} & \star & \star & \star & \dots & \star \\ 0 & 0 & \textcircled{1} & \star & \dots & \star \\ 0 & 0 & 0 & \textcircled{1} & \dots & \star \\ & & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

The leading entries are circled.

Back substitution

Suppose that we have a system of linear equations whose augmented matrix is already in row echelon form. For example, if the equations are

$$\begin{aligned} x + 2y + 5z + 6w &= -11 \\ z + 4w &= 6 \\ w &= 1 \end{aligned}$$

then the augmented matrix is

$$\left[\begin{array}{cccc|c} \textcircled{1} & 2 & 5 & 6 & -11 \\ 0 & 0 & \textcircled{1} & 4 & 6 \\ 0 & 0 & 0 & \textcircled{1} & 1 \end{array}\right]$$

Back substitution means that we use the third equation to find w, the second equation to find z and the first equation to find x. The variables whose values we find in this way (the **leading variables**) are those corresponding to the leading 1's. The other variables (called **free variables**) can take any value.

Example

Continuing with the example just given, the leading 1's are in columns 1, 3 and 4. Therefore, the leading variables are x, z and w. The only free variable is y. We set y equal to a **parameter**, which we call t.

Then from the last equation we have $w = 1$ and substituting the value of w back into the second equation we find that $z = 2$. Finally, the first equation shows that $x = -27 - 2t$. Thus the general solution is

$$\begin{aligned} x &= -27 - 2t \\ y &= t \\ z &= 2 \\ w &= 1 \end{aligned}$$

where $t \in \mathbb{R}$ is arbitrary.

Reduction to echelon form

Find the intersection of the planes
$x - 2y + 2z = -4$ *and* $2x - 3y + 5z = -1$.

This is the same example as last lecture but now we want to solve it by reducing its augmented matrix to row echelon form and then use back substitution.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 2 & -3 & 5 & -1 \end{array}\right]$$

After just one row operation we are already in row echelon form:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -4 \\ 2 & -3 & 5 & -1 \end{array}\right] \xrightarrow{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|c} \textcircled{1} & -2 & 2 & -4 \\ 0 & \textcircled{1} & 1 & 7 \end{array}\right]$$

The leading variables are x and y. The free variable is z. We put z equal to a parameter and use back substitution to solve for x and y. That is,
$x = -4t + 10$, $y = -t + 7$ and $z = t$.

Inconsistent equations

It is possible for a system to have no solutions at all. When that happens we say that the system is **inconsistent**. For example, consider

$$\begin{aligned} x + y + 2z &= 2 \\ x + 2y + 3z &= 5 \\ -x + 4y + 3z &= 8 \end{aligned}$$

We apply row operations to the augmented matrix:

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 2 \\ 1 & 2 & 3 & 5 \\ -1 & 4 & 3 & 8 \end{array}\right] \xrightarrow[R_3 := R_3 + R_1]{R_2 := R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & \textcircled{1} & 1 & 3 \\ 0 & 5 & 5 & 10 \end{array}\right]$$

$$\xrightarrow{R_3 := R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -\textcircled{5} \end{array}\right]$$

The last equation represents the equation $0 = -5$ — an impossibility. That is, there is no solution.

Types of solutions

*A system of equations in row echelon form is inconsistent if and only if there is a leading entry in the **last** column of the augmented matrix*

If the system is consistent, there is either a **unique** solution (when there are no free variables) or there are **infinitely many** solutions (when there is at least one free variable).

The number of **parameters** in the general solution is the number of **free variables**.

If for the row echelon form of the augmented matrix, there is a leading 1 in each column except the last, then the system has a **unique** solution.

Reduced row echelon form

A matrix is in **reduced row echelon form** if it is in row echelon form and, in addition, in the column which contain leading entries every other entry os 0.

We can use elementary row operations to convert a matrix in row echelon form to one in **reduced** row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 6 & -11 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 := R_1 - 6R_3]{R_2 := R_2 - 4R_3} \left[\begin{array}{cccc|c} 1 & 2 & 5 & 0 & -17 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 := R_1 - 5R_2} \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 0 & 0 & -27 \\ 0 & 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 1 \end{array}\right]$$

This is now easy to solve! In fact, the row operations we have carried out are equivalent to beck substitution.

The Gaussian elimination process

Given the augmented matrix A of a system of linear equations,

1. If all the columns are zero, then stop.

2. Find the first nonzero column.

3. Swap rows if necessary so that there is a nonzero entry a at the **top** of the first nonzero column.

4. Divide the first row by a.

5. Subtract suitable multiples of the first row from the other rows so that all entries below the leading 1 in the first row are 0.

6. If A has just one row, then stop.

7. Create a matrix B by removing the first row from A.

8. Start again at step 1 with A replaced by B.

Homogeneous equations

A system of linear equations is said to be **homogeneous** if the right hand side of every equation is 0. For example,

$$\begin{aligned} x + y + 2z &= 0 \\ x + 2y + 3z &= 0 \\ -x + 4y + 3z &= 0 \end{aligned}$$

Such a system is **always** consistent because we can always get a solution by setting all the variables to 0. Of course for some systems this will be the **only** solution but for others there can be infinitely many solutions, as in this case:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & 4 & 3 & 0 \end{array}\right] \xrightarrow[R_3 := R_3 + R_1]{R_2 := R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 := R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consistency conditions

The following example is discussed in detail in §1.5 of the printed notes.

Discuss the way the $\boldsymbol{type}$ *of the solution to the equations*

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + az &= 3 \\ x + ay + 3z &= 2 \end{aligned}$$

varies with a.

Using elementary row operations we almost convert the augmented matrix of this system to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{array}\right] \xrightarrow{\substack{\text{See} \\ \S 1.5}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & a^2+a-6 & a-2 \end{array}\right]$$

We have $a^2 + a - 6 = (a+3)(a-2)$ and so if $a = 2$ the last row is zero and there are infinitely many solutions. If $a = -3$, then the last row becomes $0\ 0\ 0 \mid -5$ and the equations are inconsistent.

Finally, if a is neither 2 nor -3, there is a unique solution.