MATH1902 Linear Algebra

Lecture 12
Week 6, Semester 1, 2001

3 April, 2001

Lecture Notes: *Linear Algebra* by R. B. Howlett
Available from Kopystop
(36 Mountain Street, Broadway)

Lecturer: Associate Professor D. E. Taylor
Room: 711, Carslaw Building
Office Hour: Tuesday 1pm – 2pm

Enquiries to: First Year Mathematics Office,
5th floor, Carslaw Building

Web:
`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

Objectives

- introduce row and column vectors
- become familiar with matrix notation
- know how to add and subtract matrices
- know how to multiply a matrix by a scalar
- know how to multiply two matrices

Row vectors

Definition A row of numbers, enclosed in parentheses, is called a **row vector**. For example, $[1, -3, 2, 0, 5, 5]$.

A row vector with just three numbers, such as $(-4, 7, 1)$ may be regarded as the coordinates of a point in space and then as the coordinates of the position vector of that point: in this case $-4\,\mathbf{i} + 7\,\mathbf{j} + \,\mathbf{k}$.

It is by analogy with this case that row vectors get their name.

We can add them and multiply them by scalars just as we do the coordinates of points in three dimensions. That is,

$$[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n] \\ = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$$

and

$$s[a_1, a_2, \ldots, a_n] = [sa_1, sa_2, \ldots, sa_n].$$

Column vectors

A row vector is just a list of numbers and instead of writing it horizontally we can write it vertically and call it a **column vector**. For example

$$\begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

Given a row vector and a column vector we can **multiply** them by multiplying corresponding entries and then adding the products; for example

$$\begin{aligned} [4, -1, 5] \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} &= 4 \times 1 + (-1) \times (-3) + 5 \times 2 \\ &= 17 \end{aligned}$$

Multiplication of row and column vectors

In general if we have a row vector and a column vector of the same length, then we define their **product** to be

$$[a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

We can use this definition to write the equations

$$\begin{aligned} 2x + 3y + 5z &= 6 \\ x + 5y - 8z &= 2 \end{aligned}$$

in the form

$$[2, 3, 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 6 \quad \text{and}$$

$$[1, 5, -8] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2.$$

Matrix example

This can be expressed more compactly as

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 5 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

and we say that the **coefficient matrix**

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 5 & -8 \end{bmatrix}$$

has been multiplied by the **column vector**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to produce the column vector

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

Matrices

In general an $r \times n$ **matrix** is a collection of rn numbers arranged in a rectangular array of r rows and n columns.

For example, $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 5 & -8 \end{bmatrix}$ is a 2×3 matrix.

The number in the i-th row and the j-th column is called the (i, j) **entry** of the matrix. For example, the $(2, 3)$ entry of the matrix above is -8.

We can think of an $r \times n$ matrix either as a vertical list of r row vectors or as a horizontal list of n column vectors.

The set of **all** $r \times n$ matrices with real number entries is denoted by $\mathbb{R}^{r \times n}$.

Matrices: addition and multiplication by scalars

If A and B are $r \times n$ matrices, then $A + B$ is defined so that the (i, j)-th entry of $A + B$ is the sum of the (i, j)-th entries of A and B. That is, if the (i, j)-th entry of A is a_{ij} and the (i, j)-th entry of B is b_{ij}, then the (i, j)-th entry of $A + B$ is $a_{ij} + b_{ij}$. For example,

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 6 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 8 \\ -3 & 4 & 7 \end{bmatrix}.$$

Similarly, multiplication by a scalar s is carried out entry by entry. That is the (i, j)-th entry of sA is sa_{ij}. For example,

$$5 \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & -15 & 10 \\ 0 & 10 & 30 \end{bmatrix}.$$

Matrices: properties of addition and multiplication by scalars

The $r \times n$ **zero matrix** is the $r \times n$ matrix $\mathbf{O}_{r\times n}$, every entry of which is 0.

To every $r \times n$ matrix A there is an $r \times n$ matrix $-A$ such that $A + (-A) = \mathbf{O}_{r\times n}$. (The entries of $-A$ are just the negative of the entries of A.)

In addition, for $r \times n$ matrices A, B and C we have

$$A + \mathbf{O}_{r\times n} = A = \mathbf{O}_{r\times n} + A$$

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$s(A + B) = sA + sB$$

$$(s + t)A = sA + tA$$

$$s(tA) = (st)A$$

$$1\,A = A$$

Note that these are **exactly** the same properties that hold for addition and scalar multiplication of vectors.

Matrices: multiplication

If A is an $r \times n$ matrix and if B is an $n \times p$ matrix, then the **product** of A and B (in that order) is the $r \times p$ matrix AB whose (i, j)-th entry is the product of the i-th row of A by the j-th column of B.

That is, if the (i, j)-th entry of A is a_{ij} and the (i, j)-th entry of B is b_{ij}, then the (i, j)-th entry of AB is

$$[a_{i1}, a_{i2}, \ldots, a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$
$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

At this point it is very convenient to introduce **sigma notation** for a sum of terms and write the (i, j)-th entry of AB as

$$\sum_{h=1}^{n} a_{ih}b_{hj}.$$

