MATH1902 Linear Algebra

Lecture 13
Week 7, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- use sigma ($\sum$) notation (also called the **summation** notation) to prove properties of matrix sums and products.

- understand how matrix multiplication relates to substitution of variables.

The associative law . . .

Given an $r \times n$ matrix A, an $n \times p$ matrix B and a $p \times q$ matrix C, we shall prove that $A(BC) = (AB)C$ by writing down the expression for the (i, j)-th entry of each product.

Let the (i, j)-th entry of A be a_{ij}, let the (i, j)-th entry of B be b_{ij} and let the (i, j)-th entry of C be c_{ij}. Then from the definition of matrix multiplication, the (j, k)-th entry of BC is the product of the j-th row of B times the k-th column of C; that is,

$$
\begin{aligned}
[b_{j1}, b_{j2}, \ldots, b_{jp}] \begin{bmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{pk} \end{bmatrix}
&= b_{j1}c_{1k} + b_{j2}c_{2k} + \cdots + b_{jp}c_{pk} \\
&= \sum_{h=1}^{p} b_{jh}c_{hk}.
\end{aligned}
$$

. . . The associative law . . .

Next, the (i, k)-th entry of $A(BC)$ is the product of the i-th row of A by the k-th column of BC and we have just seen that the j-th entry of this column is (in $\sum$ notation) $\sum_{h=1}^{p} b_{jh}c_{hk}$.

Therefore, the (i, k)-th entry of $A(BC)$ is

$$\sum_{j=1}^{n} a_{ij}(\sum_{h=1}^{p} b_{jh}c_{hk}).$$

Because of the commutative law (of addition) and distributive laws for real numbers we can write this as

$$\sum_{h=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jh}c_{hk}. \tag{1}$$

. . . The associative law . . .

As an example, take $r = n = p = 2$ and calculate the $(1, 2)$ entry of $A(BC)$. That is, $i = 1$, $k = 2$ and the entry is

$$\sum_{j=1}^{2} a_{1j}(\sum_{h=1}^{2} b_{jh}c_{hk}) =$$

$$a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22})$$

We can expand this and rearrange the terms to get

$$a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22}$$
$$= \sum_{h=1}^{2} \sum_{j=1}^{2} a_{1j}b_{jh}c_{h2}.$$

The next step is to work out the (i, k)-th entry of $(AB)C$.

. . . The associative law . . .

From its definition, the (i,h)-th entry of AB is the product of row i of A by column h of B; that is

$$[a_{i1}, a_{i2}, \ldots, a_{in}] \begin{bmatrix} b_{1h} \\ b_{2h} \\ \vdots \\ b_{nh} \end{bmatrix}$$

$$= a_{i1}b_{1h} + a_{i2}b_{2h} + \cdots + a_{in}b_{nh} = \sum_{j=1}^{n} a_{ij}b_{jh}$$

The (i,k)-th entry of $(AB)C$ is the product of the i-th row of AB by the k-th column of C and we have just seen that the h-th entry of the i-th row of AB is (in $\sum$ notation) $\sum_{j=1}^{n} a_{ij}b_{jh}$.

Therefore, the (i,k)-th entry of $A(BC)$ is

$$\sum_{h=1}^{p}(\sum_{j=1}^{n} a_{ij}b_{jh})c_{hk} = \sum_{h=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jh}c_{hk}. \qquad (2)$$

Comparing equations 1 and 2 we have the **associative law** $A(BC) = (AB)C$.

The distributive laws

There are two distributive laws: a left and a right

- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$.

We shall prove the first one and leave the second to the practice classes.

The left distributive law

Suppose that A is an $r \times n$ matrix, that B and C are both $n \times p$ matrices and use the $\sum$-notation when describing the entries of products.

The (j,k)-th entry of $B+C$ is $b_{jk}+c_{jk}$ and therefore the (i,k)-th entry of $A(B+C)$ is

$$\sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk}) = \sum_{j=1}^{n}(a_{ij}b_{jk}+a_{ij}c_{jk}).$$

On the other hand the (i,k)-th entry of AB is $\sum_{j=1}^{n} a_{ij}b_{jk}$ and the (i,k)-th entry of AC is $\sum_{j=1}^{n} a_{ij}c_{jk}$. Thus the (i,k)-th entry of $AB+AC$ is

$$\sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk} = \sum_{j=1}^{n}(a_{ij}b_{jk}+a_{ij}c_{jk}).$$

Comparing this with the previous expression we see that $A(B+C) = AB+AC$.

The failure of the commutative law

It is **not true** that $AB = BA$. In fact it may happen that only one of AB and BA is defined. For example, if $A = [2, 3]$ and $B = \left[\begin{smallmatrix} 2 & 5 \\ 7 & -1 \end{smallmatrix}\right]$.

But even if both products exist and have the same shape, they need not be equal. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Matrix multiplication and scalar multiplication

There are just two laws we need to consider. If A is an $r \times n$ matrix, B is an $n \times p$ matrix and s is a real number, then

- $(sA)B = s(AB) = A(sB)$
- $I_r A = A = A I_n$,

where I_r is the $r \times r$ **identity matrix**: every entry of I_r is 0 except for the entries down the diagonal, which are all 1. For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Substitution of variables

Suppose that we have variables x_1, x_2, . . . , x_p and variables y_1, y_2, . . . , y_n related as follows:

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1p}x_p &= y_1 \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2p}x_p &= y_2 \\ &\vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{np}x_p &= y_n. \end{aligned}$$

The coefficient matrix of this linear system is the $n \times p$ matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

Suppose that we have a third collection of variables z_1, z_2, . . . , z_r which are related to y_1, y_2, . . . , y_n by means of the linear equations:

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= z_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= z_2 \\ &\vdots \\ a_{r1}y_1 + a_{r2}y_2 + \cdots + a_{rn}y_n &= z_r. \end{aligned}$$

This time the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{bmatrix}$$

is an $r \times n$ matrix.

If we use the column vectors X, Y and Z defined by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ and } Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix},$$

then the systems of linear equations can be written $BX = Y$ and $AY = Z$.

On substituting the value of Y into the second matrix equation we find that $ABX = Z$. That is, the coefficient matrix of the equations relating the variables x_i to the variables z_j is the product AB.

See page 40 of the Linear Algebra notes for further details.