

# MATH1902 Linear Algebra

Lecture 14  
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Lecture Notes: *Linear Algebra* by R. B. Howlett  
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# Objectives

To become familiar with

- the identity matrix
- Kronecker's delta notation
- left and right inverses of a matrix
- the two-sided inverse of a matrix
- algebraic properties of the inverse
- the inverse and the solution to systems of linear equations

# The identity matrix

The  $n \times n$  **identity** matrix is the matrix with 1s on its main diagonal and 0s everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

In order to describe this matrix in a more compact form we use **Kronecker's delta** notation. This is the symbol  $\delta_{ij}$  defined to mean

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

We can now say that the  $(i, j)$ -th entry of  $I_n$  is  $\delta_{ij}$  and sometimes we shall write  $I_n = [\delta_{ij}]$ .

If  $A$  is an  $r \times n$  matrix, then  $I_r A = A = A I_n$ .

## Example and a proof

When  $r = 2$  and  $n = 3$  we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In general, the  $(i, j)$ -th entry of  $I_r A$  is

$$\begin{aligned} \sum_{k=1}^r \delta_{ik} a_{kj} &= \delta_{ii} a_{ij} \quad \text{since } \delta_{ik} = 0 \text{ when } k \neq i \\ &= a_{ij} \quad \text{since } \delta_{ii} = 1 \\ &= \text{the } (i, j)\text{-th entry of } A \end{aligned}$$

This proves that  $I_r A = A$ . That is,  $I_r$  is the matrix equivalent of 1.

Similarly  $\sum_j a_{ij} \delta_{jk} = a_{ik}$  and so  $A I_n = A$ .

# Inverses

If  $A$  is an  $r \times n$  matrix, a matrix  $B$  is called a **left inverse** of  $A$  if  $BA = I_n$ . In this case  $B$  must be an  $n \times r$  matrix for this to make sense.

For example, if

$$B = \begin{bmatrix} -3 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 3 \end{bmatrix},$$

then

$$BA = \begin{bmatrix} -3 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and so  $B$  is a left inverse of  $A$ .

A **right inverse** of  $A$  is a matrix  $C$  such that  $AC = I_r$ . In this case  $C$  must be an  $n \times r$  matrix.

In the example above  $A$  is a right inverse of  $B$ .

## Properties of the inverse

**Theorem.** *If  $A$  has a right inverse  $B$  and a left inverse  $C$ , then they are equal; i.e.,  $B = C$ .*

**Proof.** Suppose that  $BA = I_n$  and that  $AC = I_r$ .  
Then

$$B = B I_r = B(AC) = (BA)C = I_n C = C.$$

□

An **inverse** of a matrix  $A$  is a matrix  $B$  that is both a left and a right inverse of  $A$ . That is,  $BA = I$  and  $AB = I$ . Notice that this means both that  $B$  is the inverse of  $A$  **and** that  $A$  is the inverse of  $B$ .

**Theorem.** *A matrix  $A$  can have at most one inverse.*

**Proof.** Suppose that both  $B$  and  $C$  are inverses of  $A$ . Then  $B$  is both a left inverse and a right inverse, as is  $C$ . But then the previous theorem tells us that  $B = C$ . □

# Linear equations

A system of  $r$  linear equations in  $n$  unknowns

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n &= b_r.\end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix}$$

and if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix},$$

then the equations can be written in **matrix form** as

$$A\mathbf{x} = \mathbf{b}$$

# Inverses and linear equations

A matrix  $A$  is said to be **invertible** if it has an inverse. If  $A$  is invertible, its inverse is written  $A^{-1}$ .

**Theorem.** *If the coefficient matrix  $A$  of the system of linear equations  $A\mathbf{x} = \mathbf{b}$  is invertible, then the system has a unique solution and that solution is given by  $\mathbf{x} = A^{-1}\mathbf{b}$ .*

**Proof.** If  $A$  is invertible and if  $A\mathbf{x} = \mathbf{b}$ , then on multiplying both sides on the left by  $A^{-1}$  we have  $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$ . Therefore  $(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$  and on using the fact that  $A^{-1}A = I$  we find that  $I\mathbf{x} = A^{-1}\mathbf{b}$ ; that is,  $\mathbf{x} = A^{-1}\mathbf{b}$ .

Thus  $\mathbf{x} = A^{-1}\mathbf{b}$  is the only possible solution of  $A\mathbf{x} = \mathbf{b}$ . It really **is** a solution because

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

□

# Solutions of homogeneous equations

**Theorem.** *A homogeneous system of equations with fewer equations than unknowns has infinitely many solutions.*

(We shall use  $\mathbf{0}$  to denote a column vector consisting entirely of 0s.)

**Proof.** A homogenous system of equations has the form  $A\mathbf{x} = \mathbf{0}$ . After reducing the system to echelon form, the number of leading variables is at most the number of equations.

If the number of equations is less than the number of unknowns then there must be some free variables and so there are infinitely many solutions.

(Note that  $A\mathbf{x} = \mathbf{0}$  is always consistent because it always has  $\mathbf{x} = \mathbf{0}$  as a solution.)  $\square$

## The shape of the inverse

**Theorem.** *If the  $r \times n$  matrix  $A$  is invertible, then  $r = n$ . That is, an invertible matrix must be square.*

**Proof.** If  $r < n$ , then from the previous theorem the equations  $A\mathbf{x} = \mathbf{0}$  have infinitely many solutions. But this contradicts the earlier theorem that a system of linear equations with invertible coefficient matrix has a unique solution. Thus  $r \geq n$ .

To complete the proof note that  $B = A^{-1}$  is an  $n \times r$  matrix and its inverse is  $A$ . Consequently, the previous paragraph can be applied to  $B$  and we deduce that  $n \geq r$ . Thus we must have  $r = n$ .  $\square$

**Theorem.** *If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$ .*

**Proof.** We have

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= ((AB)B^{-1})A^{-1} = \\ &= (A(BB^{-1}))A^{-1} = (AI)A^{-1} = AA^{-1} = I.\end{aligned}$$

Similarly,  $(B^{-1}A^{-1})(AB) = I$  and this completes the proof.  $\square$

## The inverse of a $2 \times 2$ matrix

A direct calculation shows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}.$$

Similarly,

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}.$$

Therefore, the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

provided  $ad - bc \neq 0$ .

For example,

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$