MATH1902 Linear Algebra

Lecture 15
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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- finding the inverse of a square matrix using row operations
- introduction to elementary matrices
- write a matrix as a product of elementary matrices

Finding the inverse of a matrix

Suppose that $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

To find the inverse of A we want to find a matrix $X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ such that $AX = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(We also need to check that $XA = I$ but it will turn out that for square matrices $AX = I$ implies $XA = I$.)

By considering one column of X at a time we can write the matrix equation $AX = I$ as three separate matrix equations:

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Because these three systems have the same coefficient matrix we can solve all three at the same time by augmenting the coefficient matrix with **all three** right hand sides. That is, we combine them into the single **augmented matrix**

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] = [A \mid I]$$

Now we can carry out row operations to bring this augmented matrix to reduced row echelon form.

$$\xrightarrow[R_3 := R_3 - R_1]{R_2 := R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 := R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 := R_2 + R_3]{R_1 := R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 2 & -2 \\ 0 & \textcircled{1} & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 := R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -4 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = [I \mid A^{-1}].$$

Now that we have a **reduced** row echelon matrix, the solution for the first column of X is

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

and similarly the second and third columns of X are the second and third columns of the right hand portion of the reduced row echelon matrix.

That is, the inverse of A is

$$A^{-1} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Thus to find the inverse of A we begin with the augmented matrix $[A \mid I]$ and convert it to reduced row echelon form. If the reduced row echelon form is $[I \mid B]$, then B is the **inverse** of A.

Elementary matrices

An **elementary matrix** *is a matrix obtained by applying an elementary row operation to the identity matrix.*

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 := R_2 - 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 := \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Elementary matrices are invertible

Every elementary matrix has an inverse. The inverses of the matrices in the example just given are obtained by reversing the row operations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 := R_2 + 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 := 3R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Elementary matrices and row operations

If a matrix A is transformed into a matrix B by an elementary row operation, then $B = EA$, where E is the elementary matrix obtained by applying the same row operation to the identity matrix.

As we have seen, the row operation $R_2 := R_2 - 2R_3$ corresponds to the elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ and we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\xrightarrow{R_2 := R_2 - 2R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 2a_{31} & a_{22} - 2a_{32} & a_{23} - 2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = EA$$

Sequences of elementary row operations

If E is the elementary matrix corresponding to an elementary row operation, then the effect of carrying out the row operation on A is to transform A into EA.

More generally, if we carry out a sequence of row operations whose elementary matrices are E_1, E_2, . . . , E_k, then the result of applying these row operations to A is to change A into

$$E_k E_{k-1} \cdots E_2 E_1 A.$$

In particular, if the row operations convert A into the identity matrix, then we have $E_k E_{k-1} \cdots E_2 E_1 A = I$. If the **same** row operations convert the identity matrix I into B, then we have $E_k E_{k-1} \cdots E_2 E_1 I = B$ and therefore $BA = I$. This shows that

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1.$$

That is, the **inverse** of A is obtained by applying the **same** row operations to the identity matrix.

Products of elementary matrices

If we take the inverse of the preceding equation and use the facts that

$(A^{-1})^{-1} = A$ and $(MN)^{-1} = N^{-1}M^{-1}$

then

$$A = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}.$$

Since the inverse of an elementary matrix is again an elementary matrix it follows that we have succeeded in writing A as a product of elementary matrices.

Note. There may be more than one way to write A in this way because there may be more than one way to carry out the row operations.

