MATH1902 Linear Algebra

Lecture 16
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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- **prove** that multiplication by an elementary matrix produces an elementary row operation

- **prove** that a matrix that is row-equivalent to an invertible matrix is invertible

- given square matrices A and B, **prove** that $AB = I$ implies $BA = I$

More about matrix multiplication

Consider the $r \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{bmatrix}$$

and let $\mathbf{e}_i = [0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0]$ be the row vector of length r with 1 in the i-th column and 0 in every other place.

The next two lemmas are stated without proof but they follow directly from the definition of matrix multiplication.

Lemma 1. *The product*

$$\mathbf{e}_i \, A = [a_{i1} \ a_{i2} \ \dots \ a_{in}]$$

is the i-th row of A.

Lemma 2. *If E is an $s \times r$ matrix, then the i-th row of the product EA is obtained by multiplying A by the i-th row of E.*

Elementary matrices of *Type 1*

In the last lecture we stated that multiplication on the left by an elementary matrix has the same effect as carrying out the associated row operation. We shall now **prove** this.

The i-th row of the identity matrix is $\mathbf{e}_i$ and so the elementary matrix E obtained by applying the **Type 1** row operation $R_i := R_i + \lambda R_j$ has the same rows as the identity matrix except that its i-th row is $\mathbf{e}_i + \lambda \mathbf{e}_j$.

In this case Lemma 2 shows that the i-th row of EA is $(\mathbf{e}_i + \lambda \mathbf{e}_j)A$ and that for $k \neq i$, the k-th row of EA is $\mathbf{e}_k A$.

Now Lemma 1 shows that the rows of EA are the **same** as the rows of A except that row i of EA is row i of A added to λ times row j of A.

That is, EA is obtained from A by applying the row operation $R_i := R_i + \lambda R_j$ to A. This is the **same** row operation that was used to produce E.

Elementary matrices of *Type 2*

The elementary matrix E obtained by applying the **Type 2** row operation $R_i := \mu R_i$ $(\mu \neq 0)$ has the same rows as the identity matrix except that its i-th row is $\mu \mathbf{e}_i$.

In this case Lemma 2 shows that the i-th row of EA is $\mu \mathbf{e}_i A$ and that for $k \neq i$, the k-th row of EA is $\mathbf{e}_k A$.

Now Lemma 1 shows that the rows of EA are the **same** as the rows of A except that row i of EA is μ times row i of A.

That is, EA is obtained from A by applying the row operation $R_i := \mu R_i$ to A. This is the **same** row operation that was used to produce E.

Elementary matrices of *Type 3*

The elementary matrix E obtained by applying the **Type 3** row operation $R_i \leftrightarrow R_j$ $(i \neq j)$ has the same rows as the identity matrix except that its i-th row is $\mathbf{e}_j$ and its j-th row is $\mathbf{e}_i$.

In this case Lemma 2 shows that the i-th row of EA is $\mathbf{e}_j\,A$, the j-th row of EA is $\mathbf{e}_i\,A$ and that for $k \neq i, j$, the k-th row of EA is $\mathbf{e}_k A$.

Now Lemma 1 shows that the rows of EA are the **same** as the rows of A except that row i of EA is row j of A and row j of EA is row i of A.

That is, EA is obtained from A by applying the row operation $R_i \leftrightarrow R_j$ to A. This is the **same** row operation that was used to produce E.

We have now proved

Theorem. *If E is the $r \times r$ elementary matrix corresponding to a given elementary row operation, then EA is the matrix obtained by applying that same row operation to A.*

Row-equivalence

Matrices A and B are ***row-equivalent*** *if A can be transformed into B by a sequence of elementary row operations.*

That is, A and B are row-equivalent if we can write

$$B = E_k E_{k-1} \cdots E_2 E_1 \, A$$

where E_1, E_2, . . . , E_k are elementary matrices.

We can also write this as

$$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} \, B.$$

and we know that E_1^{-1}, E_2^{-1}, . . . , E_k^{-1} are also elementary matrices. It follows from this that we have proved

Theorem. *The matrix A is row equivalent to the identity matrix if and only if A is a product of elementary matrices.*

Reduced row echelon form

Theorem. *Suppose that B is an $n \times n$ matrix in reduced row echelon form. Then either B is the identity matrix or B has a row of zeros.*

Proof. Suppose that B has no row of zeros. Then each row has a leading 1 and we may suppose that the leading 1 in row i occurs in column c_i. Now B is an $n \times n$ matrix and because it is in echelon form, the column numbers c_i are strictly increasing.

Therefore $1 \leq c_1 < c_2 < \cdots < c_n \leq n$ and it follows that $c_1 = 1$, $c_2 = 2$, . . . , $c_n = n$. That is, B is the identity matrix. $\square$

A matrix with a row of zeros cannot be invertible. This is a consequence of the next lemma.

Lemma 3. *If A is an $r \times n$ matrix with a row of zeros and if B is an $n \times m$ matrix, then AB is an $r \times m$ matrix with a row of zeros.*

Proof. Suppose that every element of row i of A is 0. Then from Lemma 2 the i-th row of AB is $[0, 0, \ldots, 0]$. $\square$

Invertible matrices

Theorem. *Given an $n \times n$ matrix A the following are equivalent:*

1. *A is invertible.*
2. *A is row equivalent to the identity matrix I_n.*
3. *A is a product of elementary matrices.*

Proof. Suppose that A is invertible and that B is a reduced row echelon form of A. Then $B = E_k E_{k-1} \cdots E_2 E_1 A$ for some elementary matrices E_i. If B has a row of zeros, then we reach a contradiction because, on the one hand, $BA^{-1}E_1^{-1}E_2^{-1} \cdots E_{k-1}^{-1}E_k^{-1} = I$ and yet from Lemma 3 it has a row of zeros. The previous Theorem shows that the only possibility is that B is the identity matrix.

We have now shown that if A is invertible, then A is row equivalent to I_n. Furthermore, we already know that if A is row equivalent to I_n, then A is a product of elementary matrices. Finally, if A is a product of elementary matrices, then A is invertible because every elementary matrix is invertible. $\square$

Row equivalent invertible matrices

Theorem. *Matrices A and B are row-equivalent if and only if $B = MA$ for some invertible matrix M.*

Proof. If A and B are row equivalent, then $B = E_k E_{k-1} \cdots E_2 E_1 \, A$ for some elementary matrices E_i. Putting $M = E_k E_{k-1} \cdots E_2 E_1$ we see that M is invertible and $B = MA$.

Conversely, if $B = MA$ and M is invertible, then by the previous theorem we can write $M = E_k E_{k-1} \cdots E_2 E_1$ for some elementary matrices E_i. Thus A and B are row equivalent. □

Theorem. *Suppose that A and B are row-equivalent $n \times n$ matrices. Then A is invertible if and only if B is invertible.*

Proof. Since A and B are row equivalent we can write $B = MA$, where M is invertible. If A is invertible, then the inverse of B is $A^{-1}M^{-1}$. If B is invertible, the inverse of A is $B^{-1}M$. □

One-sided inverses of square matrices are two-sided

Theorem. *Suppose that A and B are $n \times n$ matrices. If $AB = I$, then $BA = I$.*

Proof. The matrix A is row-equivalent to a matrix D in reduced row echelon form and so we may write $D = MA$, where M is invertible. But now $DB = MAB = M$ and so $DBM^{-1} = I$.

From Lemma 3 we see that if D has a row of zeros, then so does I. But this is not possible and so by a previous Theorem the reduced row echelon matrix D must be the identity. From the previous paragraph we have $B = M$ and therefore $BA = MA = I$, as required. $\square$