MATH1902 Linear Algebra

Lecture 17
Week 9, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Show that the reduced row echelon form of a matrix is unique.
- Understand the concept of the **rank** of a matrix.
- Introduce **permutations** of a finite set.
- Explain the two-row notation for a permutation.
- Introduce the **diagram** of a permutation.

The rank of a matrix

Definition. *The* rank *of a matrix A is the number of nonzero rows in the reduced row echelon matrix corresponding to A.*

In order for this definition to make sense i.e., for the rank to be **well-defined**, we need to show that for every matrix A there is just one matrix in reduced row echelon form that is row equivalent to A.

We shall check this in a moment and once we know that it is true we shall also know that row equivalent matrices have the same rank.

If A is an $n \times n$ matrix, there are two possibilities for the rank of A:

1. rank$A < n$ and A is not invertible

2. rank$A = n$ and A is row equivalent to I and hence invertible.

Uniqueness of the reduced row echelon form

Theorem. *Given an $r \times n$ matrix A there is only one reduced row echelon matrix that is row equivalent to A.*

Proof. If E is a matrix with only one column and if E is in reduced row echelon form, then there are just two possibilities for E. The entry in the first row of E is either 0 or 1 and all other entries are 0.

$$E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{or} \qquad E = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Any row operation applied to the first of these matrices leaves it unchanged and so we cannot use row operations to transform the first into the second.

This means that any matrix A with just one column is equivalent to one of the matrices above but not to both of them.

. . . the proof continues . . .

Now suppose that A is an $r \times n$ matrix with $n > 1$ and that the result we want to prove is true for all matrices with fewer than n columns. (This is our **inductive hypothesis**.)

Suppose that E and F are row equivalent to A and that E and F are both in reduced row echelon form. Let A', E' and F' be the matrices obtained by deleting the last columns from A, E and F. Then E' and F' are row equivalent to A' and moreover, E' and F' are in reduced row echelon form.

(E is in **reduced row echelon form** if in each row with a nonzero entry the leading nonzero entry is 1 and the entries above and below the 1 are all 0; in addition if $i < j$, the leading 1 in row j is to the right of the leading 1 in row i. *Deleting the last column does not change this*.)

But A' has fewer columns than A and so, by our inductive hypothesis, we must have $E' = F'$. If E does not have a leading 1 in its last column we can find a column vector $\mathbf{x}$ such that $x_n \neq 0$ and $E\mathbf{x} = \mathbf{0}$. But as F is row equivalent to E we also have $F\mathbf{x} = \mathbf{0}$ and on subtracting this from the first equation we get $(E - F)\mathbf{x} = \mathbf{0}$.

. . . the proof continues . . .

The matrix $E - F$ is zero every except perhaps in the last column. And now the equation $(E - F)\mathbf{x} = \mathbf{0}$ together with $x_n \neq 0$ forces every entry in the last column to be 0. This is because the matrix equation

$$\begin{bmatrix} 0 & 0 & \dots & 0 & e_{1n} - f_{1n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & e_{in} - f_{in} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & e_{rn} - f_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

can be written
$(e_{1n} - f_{1n})x_n = \cdots = (e_{rn} - f_{rn})x_n = 0.$

In this case, since $x_n \neq 0$, we have $e_{1n} = f_{1n}$, . . . , $e_{rn} = f_{rn}$ and so $E = F$.

Similarly, if F does not have a leading 1 in its last column we can reverse the roles if E and F and use the argument just given to show once again that $E = F$.

. . . conclusion of the proof

This leaves the case in which both E and F have a leading 1 in their last columns. The row in which this 1 occurs in E must be the row following the last nonzero row of E'. The same argument applies to F and as $E' = F'$ it follows that the last columns of E and F are the same. That is, $E = F$ in all cases, as we claimed. □

Introduction to permutations

The first part of Chapter 3 of the notes is concerned with the calculation of the **determinant** of a square matrix. This is a number $\det(A)$ with the property that $\det(A) \neq 0$ if and only if A is invertible.

But before we can give a precise definition of $\det(A)$ we shall need to review some facts about **permutations**.

The simplest way to think of a permutation of the numbers $1, 2, \ldots, n$ is as a **rearrangement** of them. For example, $3, 4, 1, 2, 6, 5$ is a permutation of $1, 2, 3, 4, 5, 6$.

A more mathematical way of proceeding is to define a permutation as a certain sort of function.

Recall that a function f from a set X to a set Y is a rule that assigns an element $f(x)$ to each element x of X.

For example, if $X = \{1, 2, 3, 4\}$ and $Y = \{6, 36, -70\}$ we might define a function by the rules $f(1) = 36$, $f(2) = 6$, $f(3) = -70$ and $f(4) = 6$.

Definition. *A* permutation *of the set* $X = \{1, 2, \ldots, n\}$ *is a function* f *from* X *to* X *such that* $f(i) = f(j)$ *if and only if* $i = j$*. This is equivalent to the requirement that the numbers* $f(1)$*,* $f(2)$*, . . . ,* $f(n)$ *coincide with the numbers* $1, 2, \ldots, n$ *in some order.*

For example, $f(1) = 3$, $f(2) = 2$, $f(3) = 4$, $f(4) = 1$ defines a permutation of $\{1, 2, 3, 4\}$.

In this part of the course we shall only need to consider permutations of the sets $\{1, 2, \ldots, n\}$.

One way to describe a permutation is to use the **two row** notation. The top row consists of the numbers 1, 2, . . . , n and the bottom row consists of the numbers $f(1)$, $f(2)$, . . . , $f(n)$.

For example, the permutation above can be written as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$.

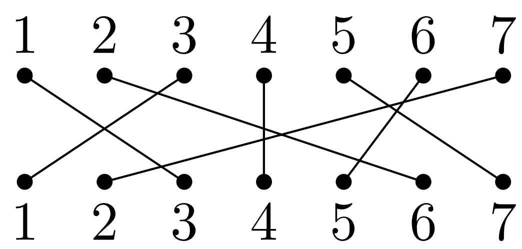
The diagram of a permutation

Given a permutation f of the set $X = \{1, 2, \ldots, n\}$ we draw the **diagram** of f by drawing two rows of n dots, both labelled 1, 2, . . . , n and then joining the dot in the top row labelled i to the dot in the bottom row labelled $f(i)$.

For example, the diagram of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 4 & 7 & 5 & 2 \end{pmatrix}$$

is



The lines in the diagram need not be straight but they must go from the top row to the bottom row without doubling back. Also, the diagram must be drawn so that at any point at most two lines cross and wherever two lines have a point in common they must cross at that point.

