

MATH1902 Linear Algebra

Lecture 18
Week 9, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Find the number of **permutations** of a finite set.
- Convert between the two-row notation for a permutation and the cycle notation.
- Find the **parity** of a permutation.
- Find the **product** of two permutations.
- Know how the parity of a product of permutations depends on the parity of the individual permutations.

Permutations

In the last lecture a permutation of the set $X = \{1, 2, \dots, n\}$ was defined to be a function f from X to X such that $f(1), f(2), \dots, f(n)$ is just a rearrangement of $1, 2, \dots, n$. We can also express this by saying that a permutation is a function $f : X \rightarrow X$ that is both **one-to-one** and **onto**.

In general, a function $f : X \rightarrow X$ is **one-to-one** if $i \neq j$ implies $f(i) \neq f(j)$. That is, f is one-to-one if there are no two elements of X at which f has the same value.

The function $f : X \rightarrow X$ is **onto** if for every element k in X there is some element j in X such that $f(j) = k$.

The number of permutations

Recall from last lecture that a permutation can be written using **two row** notation. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

represents the permutation σ such $\sigma(1) = 3$, $\sigma(2) = 5$, $\sigma(3) = 1$, $\sigma(4) = 2$ and $\sigma(5) = 4$.

The number of permutations of $\{1, 2, \dots, n\}$ is the number of two rowed arrays whose first line is $1, 2, \dots, n$ and whose second line is a rearrangement of these numbers.

The number of choices for the first entry in the second row is n ; and then the number of choices for the second entry is $n - 1$; and then the number of choices for the third entry is $n - 2$; and so on. That is, there are $n! = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$ ways to fill in the second row.

In other words, the number of permutations of n things is “ n **factorial**”, usually written $n!$.

The permutations of 2, 3 and 4 things

When $n = 2$ there are $2! = 2$ permutations:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

When $n = 3$ there are $3! = 3 \cdot 2 \cdot 1 = 6$ permutations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

When $n = 4$ there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations. Writing them out in two row notation takes up quite a lot of space.

The cycle form of a permutation

Given a permutation σ we can find the **cycle** of σ that contains a given number i by starting with i and repeatedly applying σ until we get back to i again. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 4 & 7 & 5 & 2 \end{pmatrix}$$

then the cycle containing 2 is $2, \sigma(2) = 6, \sigma(6) = 5, \sigma(5) = 7$. The next application of σ brings us back to 2. We write this cycle as $(2, 6, 5, 7)$. We could equally well start at another number and get the same cycle. For example, $(6, 5, 7, 2)$, $(5, 7, 2, 6)$ and $(7, 2, 6, 5)$ all represent the same cycle. Similarly, the cycle of σ starting at 1 is $(1, 3)$. And the cycle starting at 4 is just 4 itself.

Thus we can write σ as a list of **disjoint** cycles:

$$\sigma = (1, 3)(2, 6, 5, 7)(4)$$

When using this notation we generally don't bother to write down the cycles of length 1. For example, the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix}$$

is usually written as $\tau = (1, 5, 4, 2)$ rather than $\tau = (1, 5, 4, 2)(3)$.

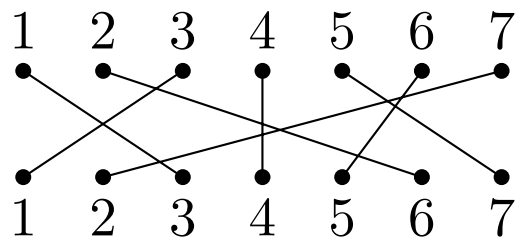
Parity

In the last lecture we saw how to associate a diagram with a permutation:

For example, the diagram of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 4 & 7 & 5 & 2 \end{pmatrix}$$

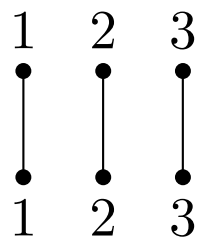
is



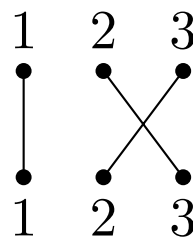
The **parity** of the permutation σ is **even** if the number of crossings is *even*; it is **odd** if the number of crossings is *odd*.

The diagram can be drawn with different numbers of crossing but whether the number is even or odd cannot be changed. We shall come back to this point later but you should convince yourself by looking at the diagrams that it is true.

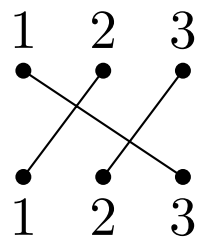
The parity of the permutations of $\{1, 2, 3\}$



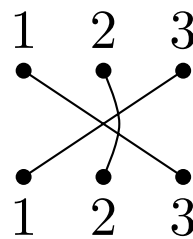
even



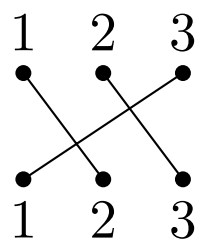
odd



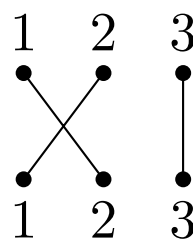
even



odd



even



odd

Composition of permutations

Given permutations σ and τ of $X = \{1, 2, \dots, n\}$, the **composite** or **product** of σ and τ (in that order) is the permutation $\sigma\tau$ defined by

$$(\sigma\tau)(i) = \sigma(\tau(i)) \quad \text{for all } i \text{ in } X.$$

For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

then

$$(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(2) = 1$$

$$(\sigma\tau)(2) = \sigma(\tau(2)) = \sigma(1) = 4$$

$$(\sigma\tau)(3) = \sigma(\tau(3)) = \sigma(4) = 3$$

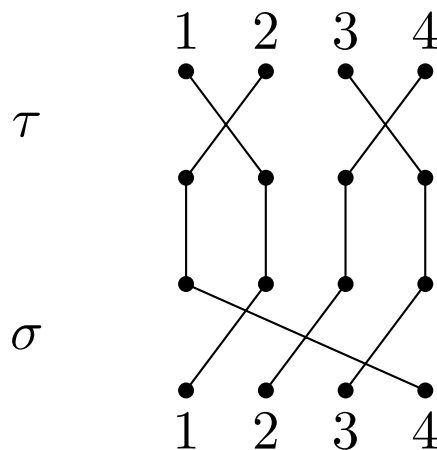
$$(\sigma\tau)(4) = \sigma(\tau(4)) = \sigma(3) = 2$$

and so

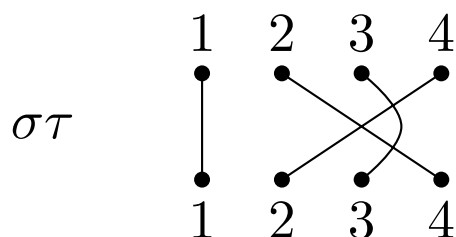
$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

The diagram of the product of two permutations

To find $\sigma\tau$ we have to apply τ *first*. Therefore the diagram of $\sigma\tau$ can be obtained by drawing the diagram of σ **underneath** the diagram of τ and joining the bottom row of dots of τ directly to the top row of dots of σ .



The diagram for $\sigma\tau$ is obtained from the diagram above by ignoring the two middle rows of dots and pulling the strings straight.



Note that $\sigma\tau \neq \tau\sigma$.

Composition and parity

The number of crossings in the diagram for $\sigma\tau$, as originally drawn, is the number of crossings in the diagram for τ plus the number of crossings in the diagram for σ .

The parity of a permutation does not depend on how the diagram is drawn and thus

the parity of $\sigma\tau$ is *even* if both σ and τ are even or if both σ and τ are odd.

the parity of $\sigma\tau$ is *odd* if σ is odd and τ is even or if σ is even and τ is odd.