MATH1902 Linear Algebra

Lecture 19
Week 10, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- The **determinant** of a 2×2 matrix
- The **minors** of a matrix
- The **cofactors** of a determinant
- The row expansion of a general determinant
- The **adjoint** matrix

2×2 **determinants**

In a previous lecture we saw that the **inverse** of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided $ad - bc \neq 0$.

The number $ad - bc$ is called the **determinant** of A and written $\det A$ of $|A|$.

For example,

$$\begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} = 4 \times 2 - 5 \times (-1) = 13.$$

Minors

Given a 3×3 matrix A, the matrix obtained by deleting the i-th row and j-th column of A is called the (i,j)-th **minor** of A; this term is also used to refer the determinants of these smaller matrices. For example, if

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 4 & 5 \\ 3 & 1 & -1 \end{bmatrix},$$

the minors of A are

$$A_{11} = \begin{bmatrix} 4 & 5 \\ 1 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -3 & 7 \\ 1 & -1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \\ 3 & -1 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix},$$

$$A_{31} = \begin{bmatrix} -3 & 7 \\ 4 & 5 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}, \quad A_{33} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}.$$

The determinants of these minors are

$$\begin{array}{lll} |A_{11}| = -9 & |A_{12}| = -13 & |A_{13}| = -14 \\ |A_{21}| = -4 & |A_{22}| = -22 & |A_{23}| = 10 \\ |A_{31}| = -43 & |A_{32}| = 19 & |A_{33}| = -2 \end{array}$$

Cofactors

If A_{ij} is the (i,j)-th minor of A, then the number $c_{ij} = (-1)^{i+j} \det A_{ij}$ is called the (i,j)-th **cofactor** of A.

If A is an $n \times n$ matrix, then in order to calculate its cofactor we need to know how to find the determinant of an $n-1 \times n-1$ matrix.

So far all we know how to do is find the determinant of a 2×2 matrix. But this is enough to find all the cofactors of the previous example:

$$\begin{array}{lll} c_{11} = -9 & c_{12} = 13 & c_{13} = -14 \\ c_{21} = 4 & c_{22} = -22 & c_{23} = -10 \\ c_{31} = -43 & c_{32} = -19 & c_{33} = -2 \end{array}$$

The definition of a 3×3 determinant

We are now in a position to define the **determinant** of a 3×3 matrix. The idea is to multiply each element of the first row of A by its cofactor and then add up the numbers.

Suppose that the (i, j)-th entry in A is a_{ij}. In the example above this yields

$$\begin{aligned} \det A &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\ &= 1 \times (-9) + (-3) \times 13 + 7 \times (-14) \\ &= -146. \end{aligned}$$

This is called the *first row expansion* of the determinant of A.

Other row expansions

We get the same result whichever row we use. For the second row we have:

$$\begin{aligned} a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= (-2)\times 4 + 4\times(-22) + 5\times(-10) \\ &= -146 = \det A. \end{aligned}$$

and for the third row:

$$\begin{aligned} a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33} \\ &= 3\times(-43) + 1\times(-19) + (-1)\times(-2) \\ &= -146 = \det A. \end{aligned}$$

On the other hand, if we use the entries in a row of A with the cofactors of a *different* row, the result is always 0. For example, taking the first row with the cofactor of the second row gives

$$\begin{aligned} a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} \\ &= 1\times 4 + (-3)\times(-22) + 7\times(-10) \\ &= 4 + 66 - 70 \\ &= 0. \end{aligned}$$

Determinants in general

We define the **determinant** of an $n \times n$ matrix inductively as the *first row expansion*:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det A_{1j} \\ &= a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n},\end{aligned}$$

where A_{1j} is the matrix obtained by deleting the first row and the j-th column from A and $c_{1j} = (-1)^{1+j} \det A_{1j}$ is the cofactor of a_{1j}.

If we define the determinant of a 1×1 matrix $[\,a\,]$ to be just the number a itself, then this formula also works for 2×2 matrices. That is,

$$\text{The cofactors of} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{are} \quad \begin{matrix} d & -c \\ -b & a \end{matrix}$$

and so the determinant is $ad - bc$.

The adjoint matrix

The **transpose** of a matrix M is the matrix M^T obtained by swapping the rows and columns of M. For example,

$$\begin{bmatrix} 1 & 2 & -3 \\ -4 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ 2 & 0 \\ -3 & 2 \end{bmatrix}$$

The **adjoint** (also called the **adjugate**) of an $n \times n$ matrix A is the transposed matrix of cofactors of A, written $\operatorname{adj}(A)$. For example, if

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 4 & 5 \\ 3 & 1 & -1 \end{bmatrix},$$

then

$$\operatorname{adj} A = \begin{bmatrix} -9 & 4 & -43 \\ 13 & -22 & -19 \\ -14 & -10 & -2 \end{bmatrix}.$$

Determinants and adjoints

Theorem. *If A is an $n \times n$ matrix, then $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\, I_n$, where I_n is the $n \times n$ identity matrix.*

This theorem will be proved in Lecture 21.

Let's check it when $n = 2$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In this case, we see that

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and therefore

$$\begin{aligned} A(\operatorname{adj} A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= (ad - bc) I_2 \\ &= \det(A)\, I_2 \end{aligned}$$

Determinants and permutations

The definition of the determinant of a 3×3 matrix by expanding along the first row leads to the following general formula:

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
\begin{aligned}
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\end{aligned}
$$

Notice that the terms have the form $\pm a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$. Furthermore the terms which occur with a $-$ sign are precisely those for which the permutation is *odd*. We shall explore this phenomenon in great detail in a later lecture.

