MATH1902 Linear Algebra

Lecture 20
Week 10, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- The connection between the adjoint and row and column expansions

- Fundamental properties of determinants: the effect of

  1. a row of zeros
  2. swapping two rows
  3. multiplying a row by a constant
  4. adding a multiple of one row to another

- Determinants of upper and lower triangular matrices

Row and column expansions and adjoints

In the last lecture we defined the **adjoint** $\operatorname{adj}(A)$ of an $n \times n$ matrix A to be the transposed matrix of cofactors of A. We also stated the following fundamental property, which we shall prove later.

Theorem. *If A is an $n \times n$ matrix, then $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n$, where I_n is the $n \times n$ identity matrix.*

The equation $A(\operatorname{adj} A) = (\det A)I_n$ means that the i-th row expansion of A always produces the determinant and that expanding along the i-th row but using the cofactors of a *different* row always produces 0.

The equation $(\operatorname{adj} A)A = (\det A)I_n$ means that the i-th column expansion of A always produces the determinant and that expanding along the i-th column but using the cofactors of a *different* column always produces 0.

Using this fundamental property we can prove many other important results about determinants.

Row and column expansion examples

First a row expansion example:

$$\begin{vmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0.$$

And now the same example, but expanding *down* the second column:

$$\begin{vmatrix} 1 & \boxed{2} & 3 \\ 4 & \boxed{5} & 6 \\ 7 & \boxed{8} & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(36 - 42) + 5(9 - 21) - 8(6 - 12)$$

$$= 12 - 60 + 48 = 0.$$

Rows of zeros

Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Theorem*. *If the $n \times n$ matrix A has a row of zeros, then $\det A = 0$.*

Proof. Suppose that every entry in row i is 0. If we expand across the i-th row, the result is

$$0c_{i1} + 0c_{i2} + \cdots + 0c_{in} = 0,$$

where c_{ij} is the cofactor of the (i, j)-th entry. Thus the determinant is 0. $\square$

From now on the theorems, such as this one, which depend on the assumption $A\,\mathrm{adj}(A) = \det(A)\,I$ will be tagged with a star ($*$).

Swapping two rows

Theorem*. *If B is the matrix obtained from A by swapping two rows, then $\det B = -\det A$. That is, swapping two rows of a matrix multiplies its determinant by -1.*

Proof. First we check this for 2×2 matrices. In this case we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{and} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

and the result is true.

Next we suppose that our matrix A has more than two rows so that we can find a number i such that the i-th row is *not* one of the rows that is swapped. We then expand along the i-th row to get

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in},$$

where $c_{ij} = (-1)^{i+j} \det A_{ij}$ and A_{ij} is the matrix obtained from A by removing the i-th row and j-th column.

In the row expansion of B along its i-th row, each minor B_{ij} is obtained from A_{ij} by swapping two rows and so, **by induction**, we have $\det B_{ij} = -\det A_{ij}$. Thus $\det B = -\det A$. $\square$

Equal rows

Theorem*. *If the matrix* A *has two rows equal, then* $\det A = 0$.

Proof. If we swap the two rows that are equal, then the matrix A remains the same and so its determinant remains the same. But the previous theorem says that the determinant should be multiplied by -1. That is, $\det A = -\det A$ and so $2\det A = 0$. This proves that $\det A = 0$. $\square$

Multiplication by a scalar

Theorem*. *If B is the matrix obtained from A by multiplying the i-th row of A by λ, then* $\det B = \lambda \det A$.

Proof. If we expand B along its i-th row we get

$$\det B = b_{i1}c_{i1} + b_{i2}c_{i2} + \cdots + b_{in}c_{in},$$

where c_{ij} is the (i,j)-th cofactor. By assumption A and B differ only in their i-th row and so the cofactors of the elements of the i-th row are the *same* for A and B. Also, we have $b_{ij} = \lambda a_{ij}$ and so

$$\begin{aligned}\det B &= \lambda a_{i1}c_{i1} + \lambda a_{i2}c_{i2} + \cdots + \lambda a_{in}c_{in} \\ &= \lambda(a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}) \\ &= \lambda \det A.\end{aligned}$$

□

Addition of a multiple of one row to another

Theorem*. *If B is the matrix obtained from A by adding λ times the i-th row of A to the j-th row of A (where $j \neq i$), then* $\det B = \det A$.

Proof. Suppose at first that A and B have more than two rows and choose a number k not equal to i or j. If we expand B along its k-th row, then

$$\det B = \sum_{\ell=1}^{n} b_{k\ell}(-1)^{k+\ell} \det B_{k\ell},$$

where $B_{k\ell}$ is the (k, ℓ)-th minor of B. In fact $B_{k\ell}$ is obtained from $A_{k\ell}$ by adding λ times one row to another. But $A_{k\ell}$ has one less row and column than A and so, provided we know the result for smaller matrices, we have $\det B_{k\ell} = \det A_{k\ell}$. Furthermore, the k-th rows of A and B are the same and therefore $b_{k\ell} = a_{k\ell}$.

This means that the above row expansion of B is also the row expansion of A and so $\det B = \det A$.

Finally, in order to start the induction we must check directly that the result is true for 2×2 matrices. That is, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{aligned} \begin{vmatrix} a + \lambda c & b + \lambda d \\ c & d \end{vmatrix} &= (a + \lambda c)d - (b + \lambda d)c \\ &= ad + \cancel{\lambda c}d - bc - \cancel{\lambda d}c \\ &= ad - bc \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix}. \end{aligned}$$

Similarly,

$$\begin{vmatrix} a & b \\ c + \lambda a & d + \lambda b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

□

Alternative Proof

Let B be the matrix obtained from A by adding λ times the i-th row of A to the j-th row of A (where $j \neq i$).

Consider the expansion of $\det(B)$ along row j. The entries in row j of B are $a_{j1} + \lambda a_{i1}$, $a_{j2} + \lambda a_{i2}$, . . . , $a_{jn} + \lambda a_{in}$ and their cofactors c_{j1}, c_{j2}, . . . , c_{jn} are the same as the cofactors of the jth row of A because the matrices A and B have the same rows except for row j.

It follows that

$$\begin{aligned}\det(B) &= (a_{j1} + \lambda a_{i1})c_{j1} + (a_{j2} + \lambda a_{i2})c_{j2} + \cdots \\ &\qquad \cdots + (a_{jn} + \lambda a_{in})c_{jn} \\ &= (a_{j1}c_{j1} + a_{j2}c_{j2} + \cdots + a_{jn}c_{jn}) \\ &\quad + \lambda(a_{i1}c_{j1} + a_{i2}c_{j2} + \cdots + a_{in}c_{jn})\end{aligned}$$

The first bracketed expression is the expansion of $\det(A)$ along row j; the second bracketed expression is the matrix product of row i of A by column j of $\operatorname{adj}(A)$ and this is 0 because $i \neq j$. Thus $\det(B) = \det(A)$ and this proves the theorem.

Upper and lower triangular matrices

A matrix is **upper triangular** if all the entries below the main diagonal are 0.

A matrix is **lower triangular** if all the entries above the main diagonal are 0.

Theorem. *The determinant of an upper triangular or a lower triangular matrix is the product of the entries on the main diagonal.*

Proof. The result is certainly true for 1×1 matrices.

If the matrix is lower triangular, then expanding along the first row gives $\det A = a_{11} \det A_{11}$. The minor A_{11} is also lower triangular and so by induction its determinant is the product of its diagonal elements.

The same proof works for upper triangular matrices if we expand along the last row (or down the first column). $\square$

Examples

If

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

then $\det A = acf$.

If

$$A = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

then $\det A = aehj$.

Calculation of a 4×4 determinant

We can use the properties of determinants described in the previous slides to simplify the calculation of large determinants

$$\begin{vmatrix} 3 & 1 & 4 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 4 & 8 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 3 & 0 & 3 \end{vmatrix}$$

$$\text{(expand down first column)} = \begin{vmatrix} 4 & 4 & 8 \\ 0 & -2 & 1 \\ 3 & 0 & 3 \end{vmatrix}$$

$$\text{(take out common factors)} = 12 \begin{vmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\text{(use } R_1 := R_1 - R_3\text{)} \quad = 12 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\text{(expand down first column)} = 12 \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= 12(1 - (-2)) = 36$$