MATH1902 Linear Algebra

Lecture 21
Week 11, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- The standard definition of the determinant
- Transpositions and row swaps
- The row expansion formula
- Proof that $A\,(\operatorname{adj} A) = (\det A)I$.

The fundamental properties

In the last lecture we proved that for every matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- swapping two rows multiplies the determinant by -1
- adding a multiple of one row to another row does not change the determinant
- multiplying a row by a scalar multiplies the determinant by the same scalar

In proving these properties we used only the assumption that the determinant of a matrix can be calculated by expanding along any row.

In this lecture we shall give a definition of the determinant that justifies these row expansion formulas.

The determinant of a 3×3 matrix

Expanding along the first row we find that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$
$$\quad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Permutations and determinants

Notice that the six terms of the expansion all have the form $\pm a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$. In fact the permutations σ are:

$$+a_{11}a_{22}a_{33} \qquad -a_{11}a_{23}a_{32} \qquad -a_{12}a_{21}a_{33}$$

$$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}$$

$$+a_{12}a_{23}a_{31} \qquad +a_{13}a_{21}a_{32} \qquad -a_{13}a_{22}a_{31}$$

$$\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}$$

The terms with *positive* sign correspond to the *even* permutations. The terms with *negative* sign correspond to the *odd* permutations.

The definition of the determinant

The calculation of the determinant of a 3×3 matrix motivates the following definition:

Definition 1. *The* $\mathbf{sign}$ *of permutation* σ *is defined to be* 1 *if* σ *is* even *and* -1 *if* σ *is* odd*. The sign of* σ *is written* $\operatorname{sgn}(\sigma)$*.*

On page 101 of the printed notes the next definition is referred to as the *non-inductive* formula for the determinant.

Definition 2. *The* $\mathbf{determinant}$ *of the* $n \times n$ *matrix* A *whose* (i, j)*-th entry is* a_{ij} *is*

$$D(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

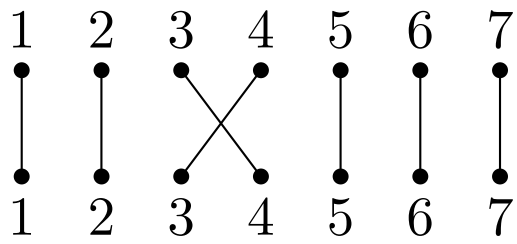
where the summation is over the $n!$ *permutations* σ *of* $\{1, 2, \ldots, n\}$*.*

We can see immediately that this definition of determinant coincides with our previous definition when n is 1, 2 or 3 and so $D(A) = \det(A)$ for $n \leq 3$. We shall soon see that $D(A) = \det(A)$ in all cases.

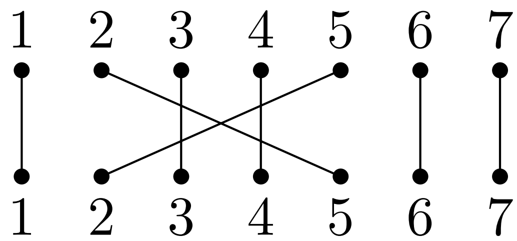
Transpositions

Definition. *A permutation that swaps two numbers and leaves all the others unchanged is called a* ***transposition****.*

For example:



and



are transpositions.

A transposition is always an *odd* permutation and so $\operatorname{sgn}(\tau) = -1$ whenever τ is a transposition.

If τ is a transposition and σ is any permutation, then $\operatorname{sgn}(\sigma\tau) = -\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau\sigma)$.

Row swaps

Using our new definition of determinant, what is the effect on the determinant of A of swapping two adjacent rows?

Suppose that rows i and $i+1$ of A are swapped to produce a new matrix B. Then the value of the determinant B, according to our new definition, is

$$
\begin{aligned}
& D(B) \\
= & \sum_\sigma \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i)} a_{i,\sigma(i+1)} \cdots a_{n\sigma(n)} \\
= & \sum_\sigma \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{i-1,\sigma(i-1)} a_{i,\sigma(i+1)} a_{i+1,\sigma(i)} \cdots a_{n\sigma(n)} \\
= & \sum_\sigma \operatorname{sgn}(\sigma) a_{1\sigma\tau(1)} \cdots a_{i-1,\sigma\tau(i-1)} a_{i,\sigma\tau(i)} a_{i+1,\sigma\tau(i+1)} \cdots a_{n\sigma\tau(n)},
\end{aligned}
$$

where τ is the transposition swapping i and $i+1$. That is, the effect of swapping two rows is to apply a transposition to every permutation in the formula for the determinant.

We know that $\operatorname{sgn}(\sigma\tau) = -1$ and that as σ runs through all the permutations of $\{1, 2, \ldots, n\}$, so does $\sigma\tau$. Thus $D(B) = -D(A)$.

Expansion along the first row

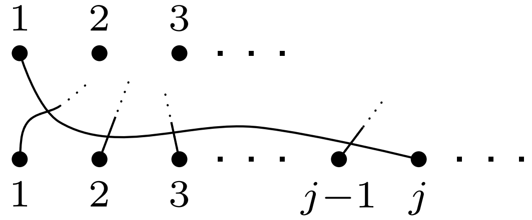
In the definition of the determinant, the coefficient of a_{1j} is

$$\sum_{\sigma} \operatorname{sgn}(\sigma) a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where the summation is over all those permutations σ such that $\sigma(1) = j$.

Up to a sign, these are precisely the terms that occur in the determinant of the $(1, j)$-th minor A_{1j} obtained by removing the first row and j-th column from A.

A permutation σ such that $\sigma(1) = j$ can be converted to a permutation of $\{1, 2 \ldots, n-1\}$ by removing the string that goes from 1 to j and then relabelling the rows of σ.



The effect of this is to change the sign of σ to $(-1)^{j-1} \operatorname{sgn}(\sigma)$.

Thus the coefficient of a_{1j} is $(-1)^{1+j} D(A_{1j})$.

Expansion along the first row

We have already observed that $D(A) = \det(A)$ when A is an $n \times n$ matrix with $n \leq 3$. Thus by induction we have

$$\begin{aligned} D(A) &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} D(A_{1j}) \\ &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det A_{1j} \\ &= \det A \end{aligned}$$

The quantity $(-1)^{1+j} \det A_{1j}$ is the **cofactor** of a_{1j} and the formula above is just the expansion along the first row, as we defined it before.

Expansion along the i-th row

To see that $\det A$ can also be calculated by the expansion along the i-th row we observe that the i-th row can be moved into first place by successively swapping it with the row above until it reaches the top of the matrix.

Each swap changes the determinant by -1 and there are $i-1$ swaps. This means that the value of the determinant changes by a factor of $(-1)^{i-1}$. We can now expand along this new first row to find that

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}.$$

This is what we have called the expansion along the i-th row and this completes the justification of our previous row expansion calculations.

The adjoint again

Consider the sum

$$\sum_{j=1}^{n} a_{kj}(-1)^{i+j} \det A_{ij}.$$

This is the (k, i)-th entry in the product $A(\operatorname{adj} A)$. If $k = i$ we have just shown that this quantity is $\det(A)$ and if $k \neq i$ it is the expansion along the i-th row of a matrix in which the i-th row and k-row are equal.

If two rows of a matrix M are equal then we know that swapping these rows changes the sign of the determinant but leaves the matrices unchanged. That is, $D(M) = -D(M)$ and so $\det(M) = D(M) = 0$. This proves that for $k \neq i$, the (k, i)-th entry in $A(\operatorname{adj} A)$ is 0. This completes the proof that

$$A(\operatorname{adj} A) = (\det A)\, I_n.$$