MATH1902 Linear Algebra

Lecture 22
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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- The determinants of elementary matrices
- The determinant criterion for a matrix to be invertible
- The product formula for determinants
- The connection between the inverse and the adjoint
- Introduction to eigenvalues and eigenvectors

The effect of elementary row operations

Recall once again that for

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

- swapping two rows multiplies the determinant by -1
- adding a multiple of one row to another row does not change the determinant
- multiplying a row by a scalar multiplies the determinant by the same scalar

Elementary matrices

The elementary matrix E_1 of **Type 1** is obtained by carrying out the row operation $R_i := R_i + \lambda R_j$ on the identity matrix. Thus $\det E_1 = 1$.

The elementary matrix E_2 of **Type 2** is obtained by carrying out the row operation $R_i := \mu R_i$ $(\mu \neq 0)$ on the identity matrix. Thus $\det E_2 = \mu$.

The elementary matrix E_1 of **Type 3** is obtained by carrying out the row operation $R_i \leftrightarrow R_j$ $(i \neq j)$ on the identity matrix. Thus $\det E_3 = -1$.

We see from this that if E is any elementary matrix, then

$$\boxed{\det(EA) = \det(E)\det(A).}$$

We can extend this to a *product* of elementary matrices:

$$\begin{aligned}
&\det(E_1 E_2 \cdots E_r A) \\
&\qquad = \det(E_1)\det(E_2 \cdots E_r A) \\
&\qquad\quad \cdots \\
&\qquad = \det(E_1)\det(E_2)\cdots\det(E_r A) \\
&\qquad = \det(E_1)\det(E_2)\cdots\det(E_r)\det(A)
\end{aligned}$$

Inverses and determinants

Theorem. *The $n \times n$ matrix A has an inverse if and only if $\det A \neq 0$.*

Proof. Suppose that D is the reduced row echelon form of A. Then $D = E_k E_{k-1} \cdots E_2 E_1 A$, for some elementary matrices E_i.

Then either D is the identity matrix, in which case A is invertible, or D has a row of zeros, in which case A is not invertible.

Taking determinants we have

$$\det(D) = \det(E_k)\det(E_{k-1})\cdots\det(E_1)\det(A)$$

and so $\det(D) = 0$ if and only if $\det(A) = 0$. If D is the identity matrix, then $\det(D) = 1$, otherwise D has a row of zeros and $\det(D) = 0$. Combining this with the previous paragraph we have $\det(A) \neq 0$ if and only if A is invertible. $\square$

Determinants and products of square matrices

Theorem. *Suppose that A and B are $n \times n$ matrices. Then*

$$\boxed{\det(AB) = \det(A)\det(B).}$$

Proof. If $\det(A) \neq 0$, then A is invertible and there are elementary matrices E_i such that

$$E_k E_{k-1} \cdots E_2 E_1 \, A = I. \qquad (1)$$

On taking determinants we have

$$\det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A) = 1. \qquad (2)$$

Multiplying equation (1) by B we have $E_k E_{k-1} \cdots E_2 E_1 \, AB = B$ and taking determinants we have $\det(E_k) \cdots \det(E_1)\det(AB) = \det(B)$. That is, $\det(AB) = \det(A)\det(B)$.

This leaves the possibility that $\det(A) = 0$. In this case AB is not invertible, otherwise we would have $AB(AB)^{-1} = I$ and A would be invertible. It follows that $\det(AB) = 0$ and therefore once again we have $\det(AB) = \det(A)\det(B)$. $\square$

The inverse and the adjoint

Recall that the **adjoint** $\operatorname{adj} A$ of a square matrix A is the transposed matrix of cofactors of A.

If $\det(A) \neq 0$, the equation $A(\operatorname{adj} A) = \det(A)\, I$ shows that the inverse of A is given by the formula

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A.$$

Eigenvalues and eigenvectors

Suppose that we have a square matrix A. In many of the applications of matrix theory it turns out to be important to find non-zero column vectors $\mathbf{v}$ and scalars λ (which may be 0) such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

In such a case the column vector $\mathbf{v}$ is called an **eigenvector** of A and the scalar λ is called the **eigenvalue** of A for $\mathbf{v}$.

Notice that the zero vector always satisfies the above equation (for any λ) but we do not count it as an eigenvector of A.

Example

Find all the eigenvectors and eigenvalues of

$$\begin{bmatrix} -1 & 3 & 1 \\ 1 & 5 & 5 \\ 0 & -3 & -2 \end{bmatrix}.$$

That is, solve the matrix equation

$$\begin{bmatrix} -1 & 3 & 1 \\ 1 & 5 & 5 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

When we write this out in full it becomes

$$\begin{aligned} -x + 3y + z &= \lambda x \\ x + 5y + 5z &= \lambda y \\ -3y - 2z &= \lambda z \end{aligned}$$

The augmented matrix for these equations is therefore

$$\left[\begin{array}{ccc|c} -1-\lambda & 3 & 1 & 0 \\ 1 & 5-\lambda & 5 & 0 \\ 0 & -3 & -2-\lambda & 0 \end{array}\right]$$

To solve these equations we can use elementary row operations to reduce the matrix to an echelon form.

The right hand side of the augmented matrix is a column of zeros. Row operations cannot change this and so there is no need to write it down again and again.

$$\begin{bmatrix} -1-\lambda & 3 & 1 \\ 1 & 5-\lambda & 5 \\ 0 & -3 & -2-\lambda \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5-\lambda & 5 \\ -1-\lambda & 3 & 1 \\ 0 & -3 & -2-\lambda \end{bmatrix}$$

$$\xrightarrow{R_2 := R_2 + (1+\lambda) R_1} \begin{bmatrix} 1 & 5-\lambda & 5 \\ 0 & 8+4\lambda-\lambda^2 & 6+5\lambda \\ 0 & -3 & -2-\lambda \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 5-\lambda & 5 \\ 0 & -3 & -2-\lambda \\ 0 & 8+4\lambda-\lambda^2 & 6+5\lambda \end{bmatrix}$$

$$\xrightarrow{R_2 := -\frac{1}{3} R_2} \begin{bmatrix} 1 & 5-\lambda & 5 \\ 0 & 1 & \frac{1}{3}(2+\lambda) \\ 0 & 8+4\lambda-\lambda^2 & 6+5\lambda \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 - (8+4\lambda-\lambda^2) R_2} \begin{bmatrix} 1 & 5-\lambda & 5 \\ 0 & 1 & \frac{1}{3}(2+\lambda) \\ 0 & 0 & \lambda^3 - 2\lambda^2 - \lambda + 2 \end{bmatrix}$$

In order that the equations have a non-zero solution the echelon form must have a row of zeros. That is, we must have $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$. This factorises as $(\lambda - 1)(\lambda + 1)(\lambda - 2) = 0$ and so the only solutions are $\lambda = 1, -1$ or 2. These are the **eigenvalues** of the matrix.