MATH1902 Linear Algebra

Lecture 23
Week 12, Semester 1, 2001

21 May, 2001

Lecture Notes: *Linear Algebra* by R. B. Howlett
Available from Kopystop
(36 Mountain Street, Broadway)

Lecturer: Associate Professor D. E. Taylor
Room: 711, Carslaw Building
Office Hour: Tuesday 1pm – 2pm

Enquiries to: First Year Mathematics Office,
5th floor, Carslaw Building

Web:
`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

Objectives

- Calculating eigenvalues and eigenvectors
- The characteristic equation

Review

Recall that

- for a square matrix A with n rows and n columns, the matrix equation $A\mathbf{x} = \mathbf{0}$ has a **non-zero** solution for $\mathbf{x}$ if and only if A is **not** invertible. (If A is invertible there is a *unique* solution, namely $\mathbf{x} = \mathbf{0}$).

- A square matrix A is invertible if and only of $\det A \neq 0$.

- An **eigenvalue** of A is a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some *non-zero* column vector $\mathbf{v}$.

- An **eigenvector** of A is a non-zero column vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

- Every non-zero multiple of an eigenvector $\mathbf{v}$ of A with eigenvalue λ is also an eigenvector of A with eigenvalue λ.

Determinants and eigenvalues

In the previous lecture we considered the problem of finding all the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 5 & 5 \\ 0 & -3 & -2 \end{bmatrix}.$$

That is, we solved the matrix equation

$$\begin{bmatrix} -1 & 3 & 1 \\ 1 & 5 & 5 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

When we write this out in full and collect terms on the left hand side, it becomes

$$\begin{aligned} (-1-\lambda)x + 3y + z &= 0 \\ x + (5-\lambda)y + 5z &= 0 \\ -3y + (-2-\lambda)z &= 0, \end{aligned}$$

that is

$$(A - \lambda I_3)\mathbf{v} = \mathbf{0}, \qquad \text{where} \quad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The equation $(A - \lambda I_3)\mathbf{v} = \mathbf{0}$ has a non-zero solution if and only if $(A - \lambda I_3)$ is **not** invertible. This is the case if and only if $\det(A - \lambda I_3) = 0$.

Expanding down the first column, the determinant is

$$
\begin{vmatrix} -1-\lambda & 3 & 1 \\ 1 & 5-\lambda & 5 \\ 0 & -3 & -2-\lambda \end{vmatrix}
$$

$$
\begin{aligned}
&= (-1-\lambda)\begin{vmatrix} 5-\lambda & 5 \\ -3 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ -3 & -2-\lambda \end{vmatrix} \\
&= (-1-\lambda)((5-\lambda)(-2-\lambda)+15) \\
&\qquad\quad - (3(-2-\lambda)+3) \\
&= (-1-\lambda)(-10-3\lambda+\lambda^2+15) - (-6-3\lambda+3) \\
&= -5-2\lambda+2\lambda^2-\lambda^3+3+3\lambda \\
&= -\lambda^3+2\lambda^2+\lambda-2 \\
&= -(\lambda-1)(\lambda+1)(\lambda-2)
\end{aligned}
$$

It follows that the eigenvalues of A are 1, -1 and 2. To find the eigenvectors we solve $(A - \lambda I_3)\mathbf{v} = \mathbf{0}$ for each possible eigenvalue. As a check we note that there *must* be a non-zero solution.

The eigenvectors

When $\lambda = 1$ the eigenvectors are the solutions of $(A - I_3)\mathbf{v} = \mathbf{0}$.

To solve these equations we can use elementary row operations to reduce the matrix to an echelon form.

The right hand side of the augmented matrix is a column of zeros. Row operations cannot change this and so there is no need to write it down again and again.

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & 4 & 5 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 5 \\ -2 & 3 & 1 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2 := R_2 + 2R_1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 11 & 11 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow[R_3 := -\frac{1}{3}R_3]{R_2 := \frac{1}{11}R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 - R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Putting $z = t$ we find that $y = -t$ and $x = -t$. Thus $\mathbf{v} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

When $\lambda = -1$ the eigenvectors are the solutions of $(A + I_3)\mathbf{v} = \mathbf{0}$.

Once again we use elementary row operations to solve the equations.

$$\begin{bmatrix} 0 & 3 & 1 \\ 1 & 6 & 5 \\ 0 & -3 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 6 & 5 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 + R_2} \begin{bmatrix} 1 & 6 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

To avoid fractions, put $z = 3t$. Then $y = -t$ and $x = -9t$. Thus the eigenvectors are $\mathbf{v} = t \begin{bmatrix} -9 \\ -1 \\ 3 \end{bmatrix}$, where $t \neq 0$.

Finally we have the case $\lambda = 2$.

$$\begin{bmatrix} -3 & 3 & 1 \\ 1 & 3 & 5 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 5 \\ -3 & 3 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 := R_2 + 3R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 12 & 16 \\ 0 & -3 & -4 \end{bmatrix}$$

Again to avoid fractions we put $z = 3t$ so that $y = -4t$ and $x = -3t$. Thus the eigenvectors are the non-zero multiples of $\begin{bmatrix} -3 \\ -4 \\ 3 \end{bmatrix}$.

The characteristic equation

The **characteristic equation** of a square matrix A is the equation

$$\det(A - xI) = 0$$

where x is the unknown.

For example, we found the characteristic equation of the matrix A of the previous example to be $-(x-1)(x+1)(x-2) = 0$.

The eigenvalues of A are the roots of the characteristic equation.

The expression $\det(A - xI)$ is called the **characteristic polynomial** of A.

For example, the characteristic polynomial of $\begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$ is

$$\begin{vmatrix} 1-x & 1 \\ 1 & 5-x \end{vmatrix} = (1-x)(5-x) - 1 = x^2 - 6x + 4$$