MATH1902 Linear Algebra

Lecture 24
Week 12, Semester 1, 2001

22 May, 2001

Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Linear combinations of vectors
- Linear independence
- Diagonalizing a matrix
- Sum and product of eigenvalues

Linear combinations

Given an $r \times n$ matrix A, let $\mathbf{a}_1$, $\mathbf{a}_2$, . . . , $\mathbf{a}_n$ be the columns of A. Then each $\mathbf{a}_j$ is an $r \times 1$ column vector and we can write

$$A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n].$$

A **linear combination** of the column vectors $\mathbf{a}_j$ is an expression of the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n,$$

where x_1, x_2, . . . , x_n are scalars.

Using matrix multiplication the column vector $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ can also be written as

$$\mathbf{b} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example

The equations

$$\begin{aligned} 2x + 3y + 3z &= 1 \\ 3x - 4y - 2x &= 0 \\ 2x - 2y - 3z &= 0 \end{aligned}$$

can be written in matrix form as

$$\begin{bmatrix} 2 & 3 & 3 \\ 3 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and also in a column vector form as

$$x \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + z \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This expresses $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of the column vectors

$$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

Linear independence

The column vectors $\mathbf{a}_1$, $\mathbf{a}_2$, . . . , $\mathbf{a}_n$ are said to be **linearly independent** if the only solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

is $x_1 = x_2 = \cdots = x_n = 0$. This is equivalent to saying that the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.

Thus the columns of an $n \times n$ matrix A are linearly independent if and only if A is invertible; that is, if and only if $\det A \neq 0$.

Eigenspaces

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_k$ are eigenvectors of a matrix B for the same eigenvalue λ. Then any linear combination $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k$ of these column vectors is also an eigenvector with eigenvalue λ. This is because

$$\begin{aligned} B\mathbf{v} &= B(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k) \\ &= x_1B\mathbf{v}_1 + x_2B\mathbf{v}_2 + \cdots + x_kB\mathbf{v}_k \\ &= x_1\lambda\mathbf{v}_1 + x_2\lambda\mathbf{v}_2 + \cdots + x_k\lambda\mathbf{v}_k \\ &= \lambda(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k) \\ &= \lambda\mathbf{v} \end{aligned}$$

The set of all eigenvectors of B with eigenvector λ together with the zero vector is called the λ-**eigenspace** of B. That is,

$$\{\, \mathbf{v} \mid B\mathbf{v} = \lambda\mathbf{v} \,\}.$$

Diagonalization

An $n \times n$ matrix A is **diagonalizable** if there is an invertible matrix T and scalars λ_1, λ_2, . . . , λ_n such that

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad (1)$$

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_n$ are the columns of T. Then the columns of AT are $A\mathbf{v}_1$, $A\mathbf{v}_2$, . . . , $A\mathbf{v}_n$ and the columns of $T \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$ are $\lambda_1\mathbf{v}_1$, $\lambda_2\mathbf{v}_2$, . . . , $\lambda_n\mathbf{v}_n$.

Writing equation (1) as

$$AT = TD \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

and comparing columns we see that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for $1 \leq k \leq n$.

Theorem. *The square matrix A is diagonalizable if and only if there is an invertible matrix T whose columns are eigenvectors of A.*

Proof. We have just seen that the columns of any invertible matrix T such that $T^{-1}AT$ is diagonal are eigenvectors of A.

Conversely, suppose that the columns of T are eigenvectors of A. Then using the same notation as before we see that $AT = TD$, where D is the diagonal matrix whose k-th diagonal entry is the eigenvalue corresponding to the k-th column of T.

Thus if T is invertible we can multiply the equation $AT = TD$ on the left by T^{-1} to get
$T^{-1}AT = D$. $\square$

Notice that the numbers on the diagonal of D are the eigenvalues of A.

Combining this theorem with our knowledge of linearly independent column vectors we see that an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Criterion for diagonalization

The characteristic equation of an $n \times n$ matrix A is a polynomial equation of degree n and therefore it has at most n distinct roots.

Theorem. *If the $n \times n$ matrix A has n **distinct** eigenvalues, then the corresponding eigenvectors are linearly independent.*

We won't prove this. Instead we shall give an example to show that if a matrix has fewer than n distinct eigenvalues then it need not be diagonalizable.

Another useful fact (that we won't prove) is that the number of linearly independent eigenvectors for a given eigenvalue λ is never greater than the multiplicity of λ as a root of the characteristic equation. To even express this properly we would need to introduce the idea of the **dimension** of the eigenspace of λ.

Example: non-diagonalizable

Consider the matrix

$$B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & -1 \\ 3 & 3 & 4 \end{bmatrix}.$$

The characteristic equation $\det(B - \lambda I) = 0$ is $(1-\lambda)^2(4-\lambda) = 0$. Thus the eigenvalues of B are 1 and 4.

When $\lambda = 1$ the eigenvectors are the solutions to the equation $(B - I)\mathbf{x} = \mathbf{0}$.

The eigenvalue 1 occurs with multiplicity 2 and so we expect no more than two linearly independent eigenvectors. We use row operations to find them:

$$\begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{R_3 := \frac{1}{3}R_3} \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 := R_3 - 2R_1]{R_2 := R_2 + 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $z = 0$ and we may take y to be a free variable and then $x = -y$. Therefore the eigenvectors are the non-zero multiples of $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

When $\lambda = 4$ the eigenvectors are the solutions of $(B - 4I)\mathbf{x} = \mathbf{0}$. We solve this in the same way:

$$\begin{bmatrix} -1 & 2 & 1 \\ -2 & -5 & -1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow[R_3 := R_3 + 3R_1]{R_2 := R_2 - 2R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 9 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 + R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 := -\frac{1}{3} R_2} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus z is a free variable and we get an eigenvector by setting $z = 3$; then $y = -1$ and $x = 1$. That is, the eigenvectors are the non-zero multiples of $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

We have shown that B does not have three linearly independent eigenvectors and so B cannot be diagonalized.

Example: diagonalizable

Sometimes, even though there are fewer than n eigenvalues we can still find enough eigenvectors. For example, suppose that

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & 3 & 4 \end{bmatrix}.$$

The characteristic equation $\det(A - \lambda I) = 0$ is $(1-\lambda)^2(4-\lambda) = 0$ (the same as before). Thus the eigenvalues of A are 1 and 4.

When $\lambda = 1$ the eigenvectors are the solutions to the equation $(A - I)\mathbf{x} = \mathbf{0}$. We use row operations to solve this:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow[]{\substack{R_2 := R_2 + R_1 \\ R_3 := R_3 - 3R_1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is one leading variable and **two** free variables. We put $y = s$ and $z = t$. Then $x = -s - t$ and so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus every eigenvector for $\lambda = 1$ is a linear combination of the linearly independent eigenvectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenspace for $\lambda = 4$ is the set of solutions of $(A - 4I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -2 & 1 & 1 \\ -1 & -4 & -1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow[]{\substack{R_1 := R1 - 2R_2 \\ R_3 := R_3 + 3R_2 \\ R_1 \leftrightarrow R2}} \begin{bmatrix} -1 & -4 & -1 \\ 0 & 9 & 3 \\ 0 & -9 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 + R_2} \begin{bmatrix} -1 & -4 & -1 \\ 0 & 9 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can put $z = 3t$ so that $y = -t$ and $x = t$. Thus the eigenvectors are the non-zero multiples of $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. We may take T to be

$$T = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

Then

$$
\begin{aligned}
AT &= \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 & 4 \\ 1 & 0 & -4 \\ 0 & 1 & 12 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
&= T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.
\end{aligned}
$$

I claim that the matrix T is invertible (because its columns are linearly independent) and therefore

$$
T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.
$$