MATH1902 Linear Algebra

Lecture 25
Week 13, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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Objectives

- Understand the relationship between the eigenvalues and the determinant of a matrix.
- Use eigenvalues to compute large powers of a matrix
- Matrices and recurrence relations
- Understand the Leslie population model.
- Introduce linear transformations of the plane.

Eigenvalues and the determinant

The eigenvalues of an $n \times n$ matrix A are the roots of the characteristic polynomial $\det(A - xI)$. The characteristic polynomial has degree n and therefore it has n roots (counted according to their multiplicity), some of which may be complex numbers.

Suppose that the eigenvalues are λ_1, λ_2, . . . , λ_n. Then

$$\det(A - xI) = (\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x).$$

If we put $x = 0$ in this equation we find that

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

That is, the determinant of a matrix is the product of its eigenvalues.

Large powers

Suppose that

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

and that we want to compute A^{16}. Even for a 2×2 matrix this is a rather unappealing task. We could save some arithmetic by computing A^2, then A^4, then A^8 and finally A^{16}, but there is another way.

The first step is to diagonalize A. The eigenvalues of A are the roots of $\det(A - \lambda I_2) = 0$. That is,

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = -3\lambda + \lambda^2 + 2 = (\lambda - 2)(\lambda - 1) = 0.$$

Thus the eigenvalues are 1 and 2.

To find the 1-eigenspace of A we solve $(A - I)\mathbf{v} = \mathbf{0}$. That is

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{where} \quad \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Putting $y = 1$ we see that $x = 1$ and so an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (All non-zero multiples of this vector are also eigenvectors.)

To find the 2-eigenspace of A we solve $(A - 2I)\mathbf{v} = \mathbf{0}$. That is

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $y = 1$ we see that $x = 2$ and so an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (All non-zero multiples of this vector are also eigenvectors.)

The next step is to use these eigenvectors as the columns of a matrix T:

$$T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Then

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

When we write out the powers of $T^{-1}AT$ we find that there is a lot of cancellation; for example

$$\begin{aligned}(T^{-1}AT)^3 &= T^{-1}A\cancel{T}\cancel{T}^{-1}A\cancel{T}\cancel{T}^{-1}AT \\ &= T^{-1}A^3T\end{aligned}$$

and in general we see that $(T^{-1}AT)^n = T^{-1}A^nT$.

To complete the calculation of A^{16} we note that

$$T^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{16} = \begin{bmatrix} 1^{16} & 0 \\ 0 & 2^{16} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 65536 \end{bmatrix}.$$

Thus

$$\begin{aligned} A^{16} &= T(T^{-1}A^{16}T)T^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 65536 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 131071 & -131070 \\ 65535 & -65534 \end{bmatrix} \end{aligned}$$

This example shows that when a matrix is diagonalizable it is fairly easy to find its powers.

Recurrence relations

Suppose that we have a sequence of numbers a_0, a_1, a_2, a_3, . . . where $a_0 = 0$, $a_1 = 1$ and for $n > 0$,

$$a_{n+1} = 3a_n - 2a_{n-1}.$$

This equation determines all the numbers a_n; it is called a **recurrence relation**. To see the connection with matrices we write the equations

$$\begin{aligned} a_{n+1} &= 3a_n - 2a_{n-1} \\ a_n &= 1a_n - 0a_{n-1} \end{aligned}$$

in matrix form:

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

and then

$$\begin{aligned}\begin{bmatrix}a_{n+1}\\ a_n\end{bmatrix} &= A\begin{bmatrix}a_n\\ a_{n-1}\end{bmatrix}\\ &= A^2\begin{bmatrix}a_{n-1}\\ a_{n-2}\end{bmatrix} = A^3\begin{bmatrix}a_{n-2}\\ a_{n-3}\end{bmatrix}\\ &\dots\\ &= A^n\begin{bmatrix}a_1\\ a_0\end{bmatrix} = A^n\begin{bmatrix}1\\ 0\end{bmatrix}.\end{aligned}$$

We know that

$$\begin{aligned}A^n &= T(T^{-1}A^nT)T^{-1}\\ &= \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2^n\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\\ &= \begin{bmatrix}2^{n+1}-1 & -2^{n+1}+2\\ 2^n-1 & -2^n+2\end{bmatrix}\end{aligned}$$

and so

$$\begin{bmatrix}a_{n+1}\\ a_n\end{bmatrix} = \begin{bmatrix}2^{n+1}-1\\ 2^n-1\end{bmatrix}.$$

From this we see that $a_n = 2^n - 1$.

Population models

(This is §3.5 of the lecture notes.)

Suppose that we are studying the growth of a population of animals over a number of years and that at year 0 we know the number of animals in each age range. If $x_k(t)$ is the number of animals aged between $k-1$ and k at year t, we would like a **formula** for $x_k(t)$, or at least a reasonable approximation. One rather simple model that enables us to solve this problem is to assume that a fixed proportion of each age group survive to the next year and that $x_1(t+1)$ is a linear combination of $x_1(t)$, $x_2(t)$, . . . , $x_n(t)$.

That is,

$$\begin{aligned} x_1(t+1) &= a_1x_1(t) + a_2x_2(t) + \cdots + a_nx_n(t) \\ x_2(t+1) &= b_1x_1(t) \\ x_3(t+1) &= b_2x_2(t) \\ &\vdots \\ x_n(t+1) &= b_{n-1}x_{n-1}(t) \end{aligned}$$

where b_i is the survival rate and a_i the average number of offspring for members of the i-th age group.

Matrices and population models

In terms of matrices we have

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ b_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

which can be written as $\mathbf{x}(t+1) = L\mathbf{x}(t)$. The coefficient matrix L is called the Leslie matrix.

Assuming the relationship $\mathbf{x}(t+1) = L\mathbf{x}(t)$ persists indefinitely we see that for all k,

$$\mathbf{x}(k) = L\mathbf{x}(k-1) = L^2\mathbf{x}(k-2) = \cdots = L^k\mathbf{x}(0).$$

We are interested in the behaviour of $L^k\mathbf{x}(0)$ as $k \to \infty$, since this will tell us what happens to the population in the long term.

Linear transformations

Given a 2×2 matrix A and a point P in the plane with coordinates (x, y) we can form the column vector $\mathbf{v} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ and apply A to $\mathbf{v}$ to obtain $A\mathbf{v}$. We say that A **transforms** $\mathbf{v}$ to $A\mathbf{v}$.

The transformation of $\mathbf{v}$ to $A\mathbf{v}$ is said to be a **linear transformation**. This means that it satisfies the rules

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, for all vectors $\mathbf{u}$ and $\mathbf{v}$, and
2. $A(a\mathbf{u}) = aA\mathbf{u}$ for all scalars a.

As a consequence of these rules a linear transformation takes lines to lines. To see this, suppose that $\mathbf{r}$ is the position vector of a point on the line through Q in the direction $\mathbf{v}$. If $\mathbf{u}$ is the position vector of Q, then for some t we have $\mathbf{r} = \mathbf{u} + t\mathbf{v}$. On applying A we have $A\mathbf{r} = A\mathbf{u} + tA\mathbf{v}$. Thus $A\mathbf{r}$ represents a point on the line through the point with position vector $A\mathbf{u}$ in the direction $A\mathbf{v}$.