MATH1902 Linear Algebra

Lecture 26
Week 13, Semester 1, 2001

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Lecture Notes: *Linear Algebra* by R. B. Howlett
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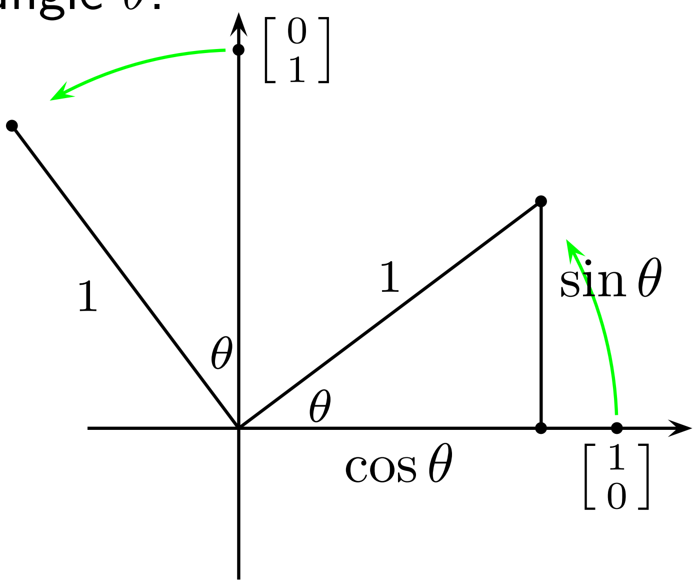
Objectives

- Be able to find the matrix representing a rotation of the plane about the origin.
- Be able to find the matrix representing a reflection.

Rotations

Let us see if we can find a matrix whose effect on the plane is to rotate each vector (at the origin) through an angle θ.



If A is a 2×2 matrix, then $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of A and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is its second column. Thus for A to represent a rotation, the first column of A must be $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and its second column must be $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. That is,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflections

What is the effect of the transformation given by the matrix

$$A = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}?$$

To find out, we first calculate its eigenvalues and eigenvectors. That is, we find λ such that $\det(A - \lambda I) = 0$. Now

$$\begin{aligned} \begin{vmatrix} -\cos\theta - \lambda & \sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} &= \lambda^2 - \cos^2\theta - \sin^2\theta \\ &= \lambda^2 - 1 \end{aligned}$$

and so $\lambda = 1$ or -1. That is, the eigenvalues are 1 and -1.

The 1-eigenspace is obtained by solving the equations

$$\begin{aligned}(-\cos\theta - 1)x + \sin\theta\, y &= 0\\ \sin\theta\, x + (\cos\theta - 1)y &= 0.\end{aligned}$$

The second equation is a multiple of the first and we can obtain a solution by putting $y = \sin\theta$ and then using the second equation to obtain $x = 1 - \cos\theta$.

The double angle formulae of trigonometry give $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $\cos\theta = 1 - 2\cos^2\frac{1}{2}\theta$. Thus

$$\begin{bmatrix} x\\ y\end{bmatrix} = 2\cos\tfrac{1}{2}\theta \begin{bmatrix}\cos\frac{1}{2}\theta\\ \sin\frac{1}{2}\theta\end{bmatrix}.$$

Similarly, to find the (-1)-eigenspace we solve

$$\begin{aligned}(-\cos\theta + 1)x + \sin\theta\, y &= 0\\ \sin\theta\, x + (\cos\theta + 1)y &= 0.\end{aligned}$$

This time we put $x = \sin\theta$ and use the first equation to obtain $y = -1 + \cos\theta$. Thus

$$\begin{bmatrix} x\\ y\end{bmatrix} = 2\cos\tfrac{1}{2}\theta \begin{bmatrix}\sin\frac{1}{2}\theta\\ -\cos\frac{1}{2}\theta\end{bmatrix}.$$

As eigenvectors for A we take the unit vectors

$$\mathbf{u} = \begin{bmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} \sin\frac{1}{2}\theta \\ -\cos\frac{1}{2}\theta \end{bmatrix}.$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, because their scalar product is 0.

The transformation given by A is the reflection in the line through the origin in the direction $\mathbf{u}$.