Lecture Notes: *Linear Algebra* by R. B. Howlett
Available from Kopystop
(36 Mountain Street, Broadway)

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Objectives

- Be able to find the matrix representing a rotation of the plane about the origin.

- Be able to find the matrix representing a reflection.
Rotations

Let us see if we can find a matrix whose effect on the plane is to rotate each vector (at the origin) through an angle $\theta$.

If $A$ is a $2 \times 2$ matrix, then $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of $A$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is its second column. Thus for $A$ to represent a rotation, the first column of $A$ must be $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and its second column must be $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$. That is,

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
Reflections

What is the effect of the transformation given by the matrix

\[ A = \begin{bmatrix} -\cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]?

To find out, we first calculate its eigenvalues and eigenvectors. That is, we find \( \lambda \) such that \( \det(A - \lambda I) = 0 \). Now

\[
\begin{vmatrix} -\cos \theta - \lambda & \sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} = \lambda^2 - \cos^2 \theta - \sin^2 \theta \\
= \lambda^2 - 1
\]

and so \( \lambda = 1 \) or \(-1\). That is, the eigenvalues are 1 and \(-1\).
The 1-eigenspace is obtained by solving the equations

\((- \cos \theta - 1)x + \sin \theta y = 0\)
\(\sin \theta x + (\cos \theta - 1)y = 0.\)

The second equation is a multiple of the first and we can obtain a solution by putting \(y = \sin \theta\) and then using the second equation to obtain \(x = 1 - \cos \theta.\)

The double angle formulae of trigonometry give
\(\sin \theta = 2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta\) and \(\cos \theta = 1 - 2 \cos^2 \frac{1}{2} \theta.\)
Thus
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = 2 \cos \frac{1}{2} \theta \begin{bmatrix}
\cos \frac{1}{2} \theta \\
\sin \frac{1}{2} \theta
\end{bmatrix}.
\]

Similarly, to find the \((-1)\)-eigenspace we solve

\((- \cos \theta + 1)x + \sin \theta y = 0\)
\(\sin \theta x + (\cos \theta + 1)y = 0.\)

This time we put \(x = \sin \theta\) and use the first equation to obtain \(y = -1 + \cos \theta.\) Thus
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = 2 \cos \frac{1}{2} \theta \begin{bmatrix}
\sin \frac{1}{2} \theta \\
- \cos \frac{1}{2} \theta
\end{bmatrix}.
\]
As eigenvectors for $A$ we take the unit vectors

$$u = \begin{bmatrix} \cos \frac{1}{2} \theta \\ \sin \frac{1}{2} \theta \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} \sin \frac{1}{2} \theta \\ - \cos \frac{1}{2} \theta \end{bmatrix}.$$  

The vectors $u$ and $v$ are perpendicular, because their scalar product is 0.

The transformation given by $A$ is the reflection in the line through the origin in the direction $u$.  