Lecture 4 MATH1904

• **Inverses**

Given a bijection \( f : A \rightarrow B \), the *inverse* of \( f \) is the bijection \( g : B \rightarrow A \) defined by \( g(b) = a \) whenever \( f(a) = b \).

The arrow diagram for \( g \) is obtained from that of \( f \) by reversing the direction of all the arrows. We often write \( f^{-1} \) to denote the inverse of \( f \).

• **The identity function**

For every set \( A \) there is a function \( I_A : A \rightarrow A \) defined by \( I_A(x) = x \). We call \( I_A \) the *identity function* for \( A \).
Composition of Functions

Given functions $f : A \to B$ and $g : B \to C$ it is possible to construct a new function $h : A \to C$ by first applying $f$ and then applying $g$. More precisely, for all $a \in A$, we define $h(a)$ by

$$h(a) = g(f(a)).$$

We write $h = g \circ f$ and call $h$ the \textit{composition} of $f$ and $g$. The arrow diagram for $g \circ f$ is obtained by merging the arrow diagrams of $f$ and $g$. That is, the head of the arrow from $a$ to $f(a)$ joins the tail of the arrow from $f(a)$ to $g(f(a))$ and we think of the combination as a single arrow from $a$ to $g(f(a))$. 
• **One-to-one functions**

The function $f : A \rightarrow B$ is one-to-one if and only if there is a function $g := B \rightarrow A$ such that $g \circ f = I_A$.

• **Onto functions**

The function $f : A \rightarrow B$ is onto if and only if there is a function $g : B \rightarrow A$ such that $f \circ g = I_B$.

• **Inverses**

The functions $f : A \rightarrow B$ and $g : B \rightarrow A$ are inverses if and only if $f \circ g = I_B$ and $g \circ f = I_A$. 
More About Permutations

A permutation is called a transposition if it interchanges just two elements and leaves the others in place. For example, the function $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ such that $f(1) = 1$, $f(2) = 5$, $f(3) = 3$, $f(4) = 4$ and $f(5) = 2$ is a transposition.

We shall show that there is a way to list all permutations of $\{1, 2, \ldots, n\}$ so that successive permutations differ only by a transposition of digits in adjacent positions.

Represent each permutation by the list of its values. In this notation the permutations of $\{1, 2, 3\}$ are

$$123 \quad 132 \quad 312 \quad 321 \quad 231 \quad 213$$
Suppose that we already have a list of all permutations of \( \{1, 2, \ldots, n-1\} \). To get the list of permutations of \( \{1, 2, \ldots, n\} \) we insert \( n \) into each of our given permutations in all possible ways, beginning at the right of the first permutation and then moving from right to left, left to right, and so on.

This process is called the Johnson-Trotter algorithm.

Here are the permutations of \( \{1, 2, 3, 4\} \) listed according to the Johnson-Trotter algorithm:

\[
\begin{align*}
1234 & \quad 1243 & \quad 1423 & \quad 4123 \\
4132 & \quad 1432 & \quad 1342 & \quad 1324 \\
3124 & \quad 3142 & \quad 3412 & \quad 4312 \\
4321 & \quad 3421 & \quad 3241 & \quad 3214 \\
2314 & \quad 2341 & \quad 2431 & \quad 4231 \\
4213 & \quad 2413 & \quad 2143 & \quad 2134
\end{align*}
\]

Note that if \( t \) is the transposition which swaps \( i \) and \( i+1 \) and if \( f \) is any permutation, then \( f \circ t \) is the permutation obtained from \( f \) by swapping the symbols in positions \( i \) and \( i+1 \).
• Parity of transpositions

Every transposition is an odd permutation.

• Parity and composition

If \( f \) and \( g \) are permutations, then the parity of \( g \circ f \) depends only on the parity of \( f \) and \( g \). If \( f \) and \( g \) are both odd or both even, then \( g \circ f \) is even. If one of \( f \) or \( g \) is odd and the other is even, then \( g \circ f \) is odd.

• Parity and transpositions

The Johnson-Trotter algorithm shows that every permutation can be obtained as a composition of transpositions and so \( f \) is an even permutation if and only if it can be written as a composition of an even number of transpositions.