Multinomial Coefficients

The number of ordered arrangements of \( n \) objects, in which there are \( k_1 \) objects of type 1, \( k_2 \) objects of type 2, \ldots, and \( k_m \) objects of type \( m \) and where

\[
k_1 + k_2 + \cdots + k_m = n,
\]

is

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}.
\]

This number is called a multinomial coefficient.

The Multinomial theorem

\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.
\]

A multinomial is an expression of the form

\[
x_1 + x_2 + \cdots + x_m.
\]
Examples

• The number of ways to arrange 5 $a$’s, 3 $b$’s and 4 $c$’s is
  \[
  \binom{12}{5,3,4} = \frac{12!}{5!3!4!}
  \]

• The number of ways to place 15 labelled balls in 5 boxes with 3 balls to each box is
  \[
  \binom{15}{3,3,3,3,3} = \frac{15!}{(3!)^5}
  \]

• The coefficient of $x_1^2x_2^3x_3^4$ in $(x_1 + x_2 + x_3)^9$ is
  \[
  \binom{9}{2,3,4} = \frac{9!}{2!3!4!}
  \]
Binomial identities

\[ 2^m = \sum_{k=0}^{m} \binom{m}{k} \] (1)

\[ \sum_{k=0}^{m} (-1)^k \binom{m}{k} = \begin{cases} 0 & m \neq 0 \\ 1 & m = 0 \end{cases} \] (2)

\[ \binom{m}{n} = 0, \quad \text{if } n > m \] (3)

\[ \binom{m}{0} = 1 \] (4)

\[ \binom{m}{k} = \binom{m}{m-k} \] (5)
\[
\binom{m}{k} = \frac{m}{k} \binom{m-1}{k-1} \tag{6}
\]

\[
\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k} \tag{7}
\]

- **Pascal’s Triangle**

\[
\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}
\]

- **The Vandermonde identity**

\[
\binom{w+m}{n} = \sum_{k=0}^{n} \binom{w}{k} \binom{m}{n-k}
\]
Introduction to the WZ method

Suppose we wish to prove an identity of the form $\sum_k f(n, k) = g(n)$, where $f(n, k)$ is constructed from binomial coefficients. For example,

$$\sum_k \binom{n}{k} = 2^n.$$

A method for doing this has been discovered by Herb Wilf and Doron Zeilberger first published in 1990 and now explained in detail in their delightful book $A = B$ with Marko Petkovšek.

The method is very versatile but it is best carried out with a computer. It is available for Mathematica and for Maple.
Here is the WZ method, taken from the book $A = B$:

1. Divide through by the right hand side to change the identity to
   \[
   \sum_k F(n, k) = 1,
   \]
   where $F(n, k) = f(n, k)/g(n)$.

2. Find a function $R(n, k)$ with the amazing property that for $G(n, k) = R(n, k)F(n, k)$ we have
   \[
   F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).
   \]

3. Sum the equation over all values of $k$ and observe that the right hand side is a “telescoping sum” which telescopes to 0. This means that
   \[
   \sum_k F(n + 1, k) = \sum_k F(n, k).
   \]

4. From the previous step we see that $\sum_k F(n, k)$ does not depend on $n$. That is, it is a constant.

5. Verify that the constant is 1 by checking that $\sum_k F(0, k) = 1$. 

6.
The identity $\sum_k \binom{n}{k} = 2^n$

We have $F(n, k) = \binom{n}{k}/2^n$ and it turns out that $R(n, k) = \frac{k}{2(k-n-1)}$.

Then $G(n, k) = R(n, k)F(n, k)$ simplifies to $-(\binom{n}{k-1})2^{-n-1}$ and we can check directly that $F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$.

That is

$\binom{n+1}{k}2^{-n-1} - \binom{n}{k}2^{-n} = -\binom{n}{k}2^{-n-1} + \binom{n}{k-1}2^{-n-1}$

Notice now that

$F(0, k) = \binom{0}{k} = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$

and therefore $\sum_k F(0, k) = 1$. 

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Example identities

\[ \sum_{k} (-1)^{k} \binom{n}{k} \frac{x}{k+x} = \binom{x+n}{n}^{-1} \]

In this case \( F(n, k) = (-1)^{k} \binom{n}{k} \frac{x}{k+x} \binom{x+n}{n} \)
and \( R(n, k) = \frac{k(k+x)}{(n+1)(k-n-1)} \).

\[ \sum_{k} \binom{n}{k} \binom{x}{k+r} = \binom{n+x}{n+r} \]

In this case \( F(n, k) = \binom{n}{k} \binom{x}{k+r} / \binom{n+x}{n+r} \) and \( R(n, k) = \frac{k(k+r)}{(n+x+1)(k-n-1)}. \)